Regulating Inaction: The Case of Price Walking*

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We develop a theoretical model and evaluate its predictions using granular search, choice, and insurer pricing data around the introduction of "price walking" regulation in the UK motor insurance market. Prior to the policy, insurers benefited from customer inaction by keeping prices low to attract new customers while raising prices for pre-existing customers. Post-regulation, introductory discounts to likely inactive customers fell, but insurers responded by proliferating products and segmenting the market more finely. The net effect is striking: Inactive customers continue to pay a substantial price penalty relative to active searchers, just through different mechanisms. These findings illustrate the difficulty of regulating markets with inertial consumers, because suppliers can redesign their product offerings in ways that undermine even well-designed interventions.

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1 Introduction

Many consumers are inert, failing to act even when the benefits of doing so are large. Firms profit from the behavior of these inert customers, who are often less educated and have lower incomes. In response, policymakers have applied a range of indirect remedies, including encouraging competition or "nudging" consumers through default options, though these remedies have sometimes delivered limited benefits for consumers. Increasingly, regulators are adopting a more muscular approach, directly regulating prices to protect customers. The goal of this paper is to investigate these issues—consumer inaction, firm behavior, and the impact of direct regulatory interventions in pricing—by studying a uniquely informative episode in the UK motor insurance market.

We study the impact of a fair pricing regulation introduced in the UK insurance market in 2022. The regulation banned "price walking," whereby insurers offer low introductory prices to attract new customers while progressively raising prices for existing customers. Because existing customers face no compulsion to search for alternative policies, they are potentially more likely to be inactive and thus face price increases. Consumer advocates recognized this dynamic and pressured regulators to act, culminating in the UK regulation, which shares a very similar design with direct price regulations recently introduced in insurance, energy, telecoms, and lending markets around the world.[1] More specifically, the regulation required firms to price consistently across their back books (i.e., the stock of existing customers) and their front books (i.e., the flow of new customers), creating a fundamental trade-off: raising prices to profit from the back book reduces their ability to attract new customers. We examine how insurers and customers navigated this trade-off and whether the regulation achieved its protective goals.

We begin by developing a theoretical framework to capture the effects of price walking regulation, modelling a market in which all consumers begin by actively choosing among firms' offered products, but a fraction becomes inert following these initial choices. Firms

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[1] Appendix Table A.1 documents that concerns about price walking and related pricing practices are prevalent around the world. In the United States, 20 states have adopted measures restricting or prohibiting price discrimination (usually referred to as price optimization or elasticity-based pricing) in personal lines, including motor insurance. For example, a 2015 task force by the National Association of Insurance Commissioners concludes that state regulators should treat pricing based on demand elasticity, propensity to shop, and retention as inconsistent with the ban on unfair discrimination (`content.naic.org/sites/default/files/inline-files/committees_c_catf_related_price_optimization_white_paper.pdf`). In the European Union, European Insurance and Occupational Pensions Authority (2023) supervisory statement emphasizes that price walking carries a high risk of unfair outcomes and is not aligned with fair-treatment obligations.

face demand shocks in each period, capturing exogenous changes to their market shares, and set initial and renewal prices to maximize profits subject to prevailing regulation. We characterize the Markov Perfect Equilibria of this model, deriving a set of testable implications about firms' endogenous responses to both the announcement and implementation of the regulation. These responses include the reduction of teaser discounts on initial prices, prices that increase in the size of insurers' back books, and market shares that oscillate through time. Taken together, the model highlights a central tension: regulations designed to protect inertial consumers can trigger supply-side responses that undermine their intended effects.

To test these predictions of the model, our analysis exploits a novel dataset on customer search, choice, and insurer pricing sourced from *Compare the Market*, the largest price comparison website in the UK motor insurance market. For 8 million insurance enquiries between 2019 and 2023, we observe the characteristics of customers that shop for car insurance, the set of different insurance products and prices that insurers offer them when they request a quote, and the product that they select.

We document five sets of facts about how insurer behavior changed following the implementation of this regulation in the UK, all of which are consistent with the model. First, insurers reduce the teaser rates they offer to customers who are more likely to be inactive. Customers who are younger, insure cheaper cars, shop for less comprehensive insurance, or are out of work due to sickness or disability are more likely to be inactive after initially searching for and taking out insurance. In the run-up to the policy, prices for these likely inactive customers rise significantly relative to those who are more likely to search.

Second, firms adjust pricing as a function of their back books. Following policy implementation, firms with products that have large back books of customers offer new, similar customers less competitive prices for these products. These effects are large: increasing a product's back book from 0 to the 99th percentile decreases the probability that the product is the cheapest quoted one for a given customer by over ten times the unconditional mean.

Third, this pricing response causes market shares to oscillate through time. Before the policy, products popular in a given customer segment tend to remain popular the following year. After implementation, this autocorrelation reverses: firms with large back books raise prices to harvest existing customers, but the regulatory pricing constraint simultaneously makes them less competitive for new ones. Thus, products gaining market share in one period become less competitive in the next.

Fourth, the regulation triggers substantial product proliferation, concentrated among the most affected firms. By introducing new products, firms can reset their back book constraints, enabling them to simultaneously harvest existing customers and compete aggressively for new ones. The top third of firms by back book size increase their product offerings by 50% following the policy, while other firms leave their menus largely unchanged.

Fifth, despite the regulation, inactive customers continue to pay a substantial price penalty. Customers who do not switch their insurance product pay 40% above the cheapest available price post-regulation, with product proliferation emerging as a key mechanism through which insurers sustain this penalty.

These findings form a coherent picture. Our model and results show that banning direct price discrimination against passive customers does not ensure good outcomes. Prior to the policy, firms price walk their back books while keeping front book prices low to attract new customers. After the policy, firms can no longer charge differential prices, but adapt in two ways: they compete less aggressively for inert customers, whose relative attractiveness has fallen, and they introduce new products to more finely segment the market. The net result is that passive customers continue to pay a substantial price penalty relative to active searchers, just through different mechanisms. These findings speak to a broader challenge in consumer protection regulation: when suppliers can endogenously respond to regulatory constraints, even well-designed interventions may fail to deliver their intended benefits.

The rest of the paper is organised as follows. The remainder of this section discusses related literature. Section 2 describes the institutional setting and the 2022 regulation. Section 3 sets up and solves the model, and discusses testable implications. Section 4 introduces our data. Section 5 describes our empirical approach to testing the model's predictions and our results, and Section 6 concludes. The appendix contains model-related technical details and proofs of the model's propositions.

1.1 Related Literature

Our paper contributes to several strands of literature. First, we contribute to the large literature studying inertia and inattention in household finance and insurance markets (e.g., Madrian and Shea, 2001; Choi, Laibson, Madrian, and Metrick, 2003; Handel, 2013; Honka, 2014; Stango and Zinman, 2016; Ho, Hogan, and Scott Morton, 2017; Allen, Clark, and Houde, 2019; Andersen, Campbell, Nielsen, and Ramadorai, 2020; Gottlieb and Smetters, 2021; Heiss, McFadden, Winter, Wuppermann, and Zhou, 2021; Brot-Goldberg, Layton, Vabson, and Wang, 2023; Fisher, Gavazza, Liu, Ramadorai, and Tripathy, 2024; Allen and

Li, 2025). Many of these papers study how inactive customers (who are often poorer and less financially sophisticated) pay higher prices than active customers for the same product, thereby benefiting firms. Handel (2013) shows that inertia and choice frictions mitigate adverse selection concerns; indeed, Chiappori and Salanié (2000) find no evidence of adverse selection in the French motor insurance market for young drivers. These findings motivate us to focus on how firms' dynamic pricing and product design aim to exploit customer inaction, even when regulation bans direct price discrimination against passive customers.

Second, we contribute to the literature on consumer protection regulations in financial product markets (Campbell, 2006; Campbell and Ramadorai, 2025; Cuesta and Sepúlveda, 2021). Agarwal, Chomsisengphet, Mahoney, and Stroebel (2015) and Nelson (2025) find that the CARD Act reduced credit card fees and interest rates, increasing consumer surplus. The regulation we study ties prices for existing customers to those for prospective customers. We show firms respond by expanding the product space and segmenting the market, such that inactive customers continue to pay more than active consumers. Alternative remedies such as encouraging competition (Hortaçsu, Madanizadeh, and Puller, 2017; MacKay and Remer, 2024) and nudges (Choi, Laibson, Cammarota, Lombardo, and Beshears, 2024) face similar challenges from endogenous supplier responses, highlighting the complexities of designing consumer protection regulation.

Third, we contribute to the literature on menu design and product proliferation. Prior papers study the proliferation of consumer options, with explanations including barriers to entry (Schmalensee, 1978), obfuscation (Ellison and Ellison, 2009), and price discrimination and screening (Rothschild and Stiglitz, 1976; Marone and Sabety, 2022). We demonstrate an alternative driver: supplier responses to regulatory pricing constraints, similar to some patterns documented in the market for Medicare Part D plans (Marzilli Ericson, 2014).

Fourth, our model contributes to the literature on behavior-based price discrimination. Armstrong (2006) and Fudenberg and Villas-Boas (2006) provide insightful surveys. Our benchmark model builds on the classic framework of Beggs and Klemperer (1992), with two new features. First, we introduce demand shocks to study the heterogeneous effect of back books on prices and market shares. Second, in line with regulatory concerns about consumer exploitation, we introduce consumers who do not anticipate the possibility of being locked into contracts due to inertia (Gabaix and Laibson, 2006; Heidhues, Koszegi, and Murooka, 2017; Grubb, 2015; Heidhues and Kőszegi, 2018)

Finally, we contribute to the empirical literature on motor insurance markets. Earlier research has mainly focused on reduced-form testing for asymmetric information, with mixed results (Puelz and Snow, 1994; Chiappori and Salanié, 2000; Dionne, Gouriéroux, and Vanasse, 2001; Abbring, Chiappori, and Pinquet, 2003). Recent structural work accounts for richer sources of heterogeneity beyond unobserved risk: Israel (2005) finds lock-in effects from consumer learning; Cohen and Einav (2007) show that unobserved risk aversion dominates unobserved risk; Honka (2014) and Honka and Chintagunta (2017) quantify substantial search and switching costs; and Rubinstein (2026) studies pass-through in commercial motor insurance. We contribute by using multi-insurer data with complete customer menus to characterize demand and supply responses, and by showing how firms dynamically adjust pricing and products to exploit inertia when regulation constrains direct price discrimination.

2 Institutional Setting: The UK Motor Insurance Market and the 2022 Price Regulation

In the UK, all vehicles must be insured by law, generating gross motor insurance premiums of £16.6 billion in 2021 (Association of British Insurers, 2021). Insurers offer three levels of coverage: third-party insurance is the legal minimum and covers damage to third parties; third-party, fire, and theft additionally insures against fire damage or theft of the insured vehicle; and comprehensive insurance further covers medical expenses and damage or theft of the vehicle's contents.

Compliance with mandatory insurance is high: The Motor Insurers' Bureau estimates that at most 4% of UK drivers are not insured.[2] This contrasts sharply with the United States, where 15-20% of drivers operate without insurance (Kluender, 2023). The near-universal coverage substantially limits adverse selection concerns on the extensive margin. Moreover, approximately 90% of consumers in our data purchase comprehensive coverage, the highest tier. This concentration in the top coverage level minimizes the cream-skimming issues that can arise when consumers sort across coverage tiers (Azevedo and Gottlieb, 2017).

Within each of the three levels of coverage described above, most insurers offer multiple insurance policies, which we henceforth refer to as insurance products. These products differ in their non-price characteristics, for example, whether insured drivers are covered when driving other cars. Table A.2 in the Appendix provides some examples of the product offerings of a major UK insurance brand.

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[2] `https://www.mib.org.uk/`

More than 70% of new customers for UK motor insurance buy their insurance through a price comparison website (Financial Conduct Authority, 2018). The remaining customers either contact insurers directly or use insurance brokers. A customer provides price comparison platforms with details on the characteristics of the drivers to be insured, the car to be insured, the drivers' driving habits, and the desired level of coverage. These intermediaries pass these details on to a set of insurers, which return quotes to the customer, presented in a menu ordered from the cheapest to the most expensive. Figure A.1 in the Appendix provides an example of how a price comparison website displays the menu to a customer.

Contracts usually last one year and must be renewed annually. When renewing, customers can (i) use a price comparison website (the primary source of our data); (ii) contact an insurer directly; or (iii) accept their existing policy's renewal offer. Renewal prices may change due to factors such as vehicle depreciation, accumulated driving experience, or changes in the customer's circumstances. However, as discussed below, before the 2022 regulation, insurers also systematically raised renewal prices as part of "price walking" strategies that exploited customer inertia.

The 2022 Pricing Regulation. "Price walking" is the term used to describe an insurer ratcheting up prices for its existing customers over time, with renewing customers charged increasingly higher prices relative to similar new customers. Price walking is common in many markets and was particularly prevalent in the UK insurance market before 2022.[3]

In January 2022, the Financial Conduct Authority (FCA) implemented rules to combat price walking. It sought to end the "customer loyalty penalty," by requiring that the price insurers offered renewing customers for a product be no higher than the "equivalent new business price," which is the price they would offer a hypothetical equivalent new customer. The rules did not provide a definition of equivalence, for example by defining a set of demographic characteristics within which insurers could not discriminate based on tenure. Instead, the regulator required insurers to attest that they were complying with "both the spirit and letter of the requirements," and committed to collecting data to monitor compliance going forward.[4] The regulations are perhaps best understood, then, as requiring that insurers amend their pricing models (whatever they may be) such that they do not discriminate based on tenure.

The regulation applies regardless of the channel through which customers purchase in-

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[3] `https://www.ft.com/content/03bde15c-323b-48eb-a392-d3a874001307`

[4] https://www.fca.org.uk/publication/policy/ps21-5.pdf

surance, covering policies sold through price comparison websites, directly by insurers, and through brokers. This policy implementation followed a few years of public consultation on pricing practices in insurance. This began in 2018 when a consumer protection body complained to UK regulators about price walking in insurance markets.[5] Regulators responded by warning insurers to address these issues, and began the research and consultation process that culminated in the rule change in January 2022 (Financial Conduct Authority, 2018). The policy could therefore be anticipated several years in advance of implementation.

3 Model

We develop a simple model to derive the implications of price walking regulation in a market where some consumers exhibit inertia.

Setup. Overlapping generations of consumers populate the market. A unit mass of new consumers joins each period and stays for two periods, purchasing a good ("insurance") in each period.

Two horizontally differentiated firms ($i = 0, 1$) compete on a Hotelling line. Firms face stochastic demand shocks that shift their locations: firm 0 moves to $\ell = \varepsilon_t^0$ and firm 1 to $\ell = 1 - \varepsilon_t^1$, where higher shocks move firms closer to the center and thus increase demand. Demand shocks have support in $[0, \bar{\varepsilon}]$ with $\bar{\varepsilon} < 1/2$, ensuring firm i remains closest to endpoint i. We assume shocks are i.i.d. across firms and time and are publicly observed before firms set prices. Let $\Delta_t := \varepsilon_t^0 - \varepsilon_t^1$ denote the shock difference.

Consumers are distributed uniformly on the unit interval with linear transportation costs. A consumer's location captures brand preferences, and we normalize marginal costs to zero (we allow for negative prices so firms can lock in consumers). Firms cannot distinguish new joiners from switchers.[6] At each date t, firm i sets a new-customer price p_t^i and a renewal price r_t^i.

Consumer behavior. All consumers actively choose a firm in their first period, selecting the firm offering the highest utility net of price and transportation costs. We assume consumers are myopic, ignoring the possibility of becoming locked in through inertia.[7] A

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[5] `https://www.gov.uk/cma-cases/loyalty-penalty-super-complaint`

[6] This assumption captures that firms cannot identify "shoppers" (who regularly seek the best deal) from consumers forced to shop due to circumstances like buying a new car.

[7] This assumption is consistent with "naive" passive consumers who believe they will shop again in the next period. That belief is correct for active consumers, so they may not be naive. In our model, it is

consumer at location ℓ who buys from firm 0 receives utility:

$$V - c|\ell - \varepsilon_t^0| - p_t^0, \tag{1}$$

where gross surplus V is large enough that everyone buys and $c > 0$ parametrizes product differentiation through transportation costs. Buying from firm 1 yields:

$$V - c|1 - \varepsilon_t^1 - \ell| - p_t^1. \tag{2}$$

Equating (1) and (2) determines the indifferent consumer's location $\ell_t^* = \frac{1+\Delta_t}{2} - \frac{p_t^0 - p_t^1}{2c}$. Using the uniform distribution, we obtain firm 0's market share among active consumers:

$$s(\mathbf{p}_t; \Delta_t) := \left[\frac{1+\Delta_t}{2} + \frac{p_t^1 - p_t^0}{2c}\right]_0^1, \tag{3}$$

where $[z]_0^1 := \max\{0, \min\{z, 1\}\}$ truncates z to $[0, 1]$, and $\mathbf{p}_t := (p_t^0, p_t^1)$.

In the second period, a fraction $\lambda \in (0, 1)$ of consumers remain with their original firm ("passive"), independent of location. A regulatory price cap $K > c$ limits prices firms can charge.[8] Without regulation, firms set the maximum renewal price: $r_t^i = K$. Under price walking regulation, firms must set the same price for joiners and renewers: $p_t^i = r_t^i$. The remaining fraction $1 - \lambda$ are "active" consumers who shop every period.

Active consumers and new joiners return to the market each period, choosing the firm offering the highest utility. Without regulation, each period sees mass 1 of new customers buying and mass $1 - \lambda$ of active consumers switching to exploit lower new-customer prices (since equilibrium requires $p_t^i \leq r_t^i = K$).

3.1 No Price Walking Regulation

Without regulation, firm i sets new-customer price p_t^i and renewal price r_t^i each period. We characterize the Markov Perfect Equilibria (MPE) of the game. Since demand shocks are the only payoff-relevant variable in the game without regulation, a Markovian state is a realization of the demand shock Δ_t. In any MPE, firms always charge passive customers the

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immaterial whether passivity is a persistent type or a transitory attention shock at renewal, as in Brot-Goldberg, Layton, Vabson, and Wang (2023).

[8] Without a price cap, firms would charge infinitely high renewal prices to passive buyers, eliminating equilibrium existence.

maximum: $r_t^i = K$.

Let $\delta \in (0,1)$ denote the firms' discount factor. Firm 0's profits from a cohort joining in period t equal:

$$\Pi_t^0 := \underbrace{s(\mathbf{p}_t, \Delta_t) p_t^0}_{\text{First period}} + \delta \left[\underbrace{\lambda s(\mathbf{p}_t, \Delta_t) K}_{\text{Second period passive}} + \underbrace{(1-\lambda)\, E_t \left[s(\mathbf{p}_{t+1}, \Delta_{t+1}) p_{t+1}^0 \right]}_{\text{Second period active}} \right]. \tag{4}$$

The first term captures first-period profits when all consumers actively choose. The second term captures second-period profits from the λ fraction of consumers who passively remain and pay K. The third term captures profits from the $1-\lambda$ fraction of active consumers who return to the market.

When setting period-t price, firm 0 faces a total demand of $2-\lambda$ (mass 1 of new joiners plus mass $1-\lambda$ of active customers from the previous period), attracting the proportion $s(\mathbf{p}_t, \Delta_t)$. In the following period, a fraction λ of these customers become passive and pay K (discounted by δ). Firm 0 maximizes:

$$\underbrace{(2-\lambda) s(\mathbf{p}_t; \Delta_t)}_{\text{Active demand at } t} \cdot p_t^0 + \delta \cdot \underbrace{\lambda s(\mathbf{p}_t; \Delta_t)}_{\text{Passive demand at } t+1} \cdot K. \tag{5}$$

Each firm's profit function is strictly concave in its own price. The MPE is interior, and first-order conditions yield:

$$p_U^0(\Delta_t) = c - \frac{\delta\lambda}{2-\lambda} K + \frac{c}{3} \Delta_t, \qquad p_U^1(\Delta_t) = c - \frac{\delta\lambda}{2-\lambda} K - \frac{c}{3} \Delta_t. \tag{6}$$

Forward-looking firms charge lower first-period prices to build market share, exploiting it in the second period by charging the maximum price K. The discount increases in the gain from exploiting passive consumers (higher λ, K, or δ). In equilibrium, all firms cut prices equally, leaving market shares unchanged. Equilibrium market shares are:[9]

$$s_U^0(\Delta_t) = \frac{1}{2} + \frac{\Delta_t}{6}, \qquad s_U^1(\Delta_t) = \frac{1}{2} - \frac{\Delta_t}{6}. \tag{7}$$

Without regulation, prices and market shares are i.i.d. across time. Thus, any serial correlation in our model stems from price walking regulation.

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[9] Since $|\Delta_t| \leq \bar{\varepsilon} < \frac{1}{2}$, market shares lie in $(0,1)$, confirming the equilibrium is interior.

3.2 Price Walking Regulation

Regulation requires firms to charge the same price to joiners and renewers, eliminating the ability to hike renewal prices to K while offering lower new-customer prices.

Let $y_t := s(\mathbf{p}_{t-1}; \Delta_{t-1}) - \frac{1}{2}$ denote firm 0's lagged market share relative to an equal market split.[10] Then, firm 0's active-customer share in period $t-1$ is $\frac{1}{2} + y_t$ and firm 1's is $\frac{1}{2} - y_t$. Since a share λ of firm 0's active customers at $t-1$ become passive at t, firm 0's demand in period t equals

$$\underbrace{(2-\lambda)s(\mathbf{p}_t; \Delta_t)}_{\text{Active at } t} + \underbrace{\lambda\left(\frac{1}{2} + y_t\right)}_{\text{Passive at } t}.$$

Under regulation, firms charge the same price to all customers, so firm 0's period-t profits equal:

$$\widehat{\Pi}^0(\mathbf{p}_t, \Delta_t, y_t) := \left[(2-\lambda)s(\mathbf{p}_t; \Delta_t) + \lambda\left(\frac{1}{2} + y_t\right)\right] p_t^0.$$

Previous prices affect current profits only through the back book sizes: $\lambda\left(\frac{1}{2} + y_t\right)$ for firm 0 and $\lambda\left(\frac{1}{2} - y_t\right)$ for firm 1. Following Maskin and Tirole (2001), we identify Markovian states in the regulated model by pairs (y_t, Δ_t) of previous-period market shares and current-period demand shocks. A Markovian strategy $p_R^i : [-\frac{1}{2}, \frac{1}{2}] \times [-\bar{\varepsilon}, \bar{\varepsilon}] \to \mathbb{R}$ specifies firm i's price in each state.

Let $b \in (0,1)$ denote the unique root of

$$\delta b^3 - 3b + \frac{2\lambda}{2-\lambda} = 0, \tag{8}$$

in the unit interval, and let

$$a := \frac{2 - \delta b^2}{2(3 - \delta b^2)} \in (\frac{1}{4}, \frac{1}{3}), \qquad m := \frac{2 + \delta b}{2 - \lambda(1-\delta)} \in (1, 2). \tag{9}$$

We impose the following restrictions on the parameter space to ensure the equilibrium is

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[10] We assume an arbitrary initial market share y_1. Extensions allowing simultaneous entry with zero initial shares for both firms yield identical results.

interior:[11]

$$\bar{\varepsilon} \leq (1-b)\left(3-\delta b^2\right), \tag{10}$$

and $K \in [\underline{K}, \bar{K}]$, where

$$\begin{aligned}
\underline{K} &:= c\left(m + a\bar{\varepsilon} + \frac{b}{2}\right), \\
\bar{K} &:= c\left\{m + 1 - (1-a)\bar{\varepsilon} - \frac{b}{2} + \frac{(2-\lambda)(4-\delta b^2)}{2\lambda}\left[\frac{1}{2} - \left(\frac{1}{2} - a\right)\bar{\varepsilon} - \frac{b}{2}\right]^2\right\}.
\end{aligned} \tag{11}$$

The proposition below characterizes the unique MPE in affine strategies.[12]

Proposition 1. *There exists an MPE in which firms choose strategies:*

$$p_R^0(y_t, \Delta_t) = c\big(m + a\Delta_t + by_t\big), \qquad p_R^1(y_t, \Delta_t) = c\big(m - a\Delta_t - by_t\big). \tag{12}$$

Moreover, this is the unique MPE in which firms use affine strategies.

The equilibrium is symmetric. Prices increase with product differentiation (c) and positive demand shocks ($a > 0$). Unlike without regulation, prices now depend on back books: firms trade off revenue from back books against competitiveness for active customers. Larger back books incentivize higher prices. The pass-through a from demand shocks to prices decreases in δ and λ, whereas the pass-through b from back books to prices increases in both. As passive consumers become more prevalent, reducing current prices to invest in future back books grows more profitable, lowering current shock sensitivity and raising back-book sensitivity.

Proposition 2. *In the unique MPE in affine strategies, market shares follow a stable, oscillating AR(1) process:* $y_{t+1} = -by_t + \left(\frac{1}{2} - a\right)\Delta_t$.

In response to demand shocks, firms adjust their prices by the factor a. After a positive demand shock $\Delta_t > 0$, firm 0 increases its price by $a\Delta_t$ whereas firm 1 decreases its price by $a\Delta_t$, affecting the price differential by $2a\Delta_t$. Since $2a\Delta_t < \Delta_t$, the price adjustment is not

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[11] Condition (10) is very mild and implies that on-path market shares are interior. For example, it always holds if $\lambda < \bar{\lambda} \approx 0.9681$. Condition (11) implies that neither price caps nor off-path truncations bind. If these conditions fail, firms may prefer to give up on new customers in some states, setting the highest possible price and focusing entirely on their back book.

[12] Appendix D contains all proofs. Appendix E extends the results for interior symmetric equilibria in polynomial strategies.

enough to completely offset the demand shock, so firm 0 gains market share. This builds a larger back book, which the firm exploits by charging higher prices next period (the effect is symmetric for negative shocks). Market shares oscillate as firms with large back books charge higher prices, reducing future back books. Oscillations intensify when firms are more patient (δ high) or face more passive consumers (λ high).

3.3 Comparing Markets

We now turn to the effect of the regulation on prices and welfare. Define the average price of a new policy in the equilibrium with and without regulation as

$$\bar{p}_R := \frac{p_R^0(y_t, \Delta_t) + p_R^1(y_t, \Delta_t)}{2} \quad \text{and} \quad \bar{p}_U := \frac{p_U^0(\Delta_t) + p_U^1(\Delta_t)}{2}, \tag{13}$$

which, by equations (6) and (12), are constant across states.

Proposition 3. *In the unique MPE in affine strategies, the average price of a new policy is higher under regulation: $\bar{p}_R > \bar{p}_U$. Moreover, the price increase $\bar{p}_R - \bar{p}_U$ is increasing in consumer inertia (λ).*

The model predicts that new-policy prices increase following regulation. Without regulation, firms offer discounts to attract passive customers, then charge maximum price K at renewal. Regulation reduces renewal prices, eliminating discount incentives. Price increases are larger for more passive groups because discounts to attract them are more profitable.

We now turn to the effects of regulation on welfare. Since all consumers purchase coverage, total surplus depends only on allocative efficiency (transportation costs). Price differences merely transfer surplus between customers and firms. Total surplus maximization requires each customer to buy from the closest firm, achieved when firms charge identical prices. Neither regime achieves this.

Without regulation, firms increase prices in response to demand shocks, extracting surplus from inframarginal consumers at the expense of losing sales from marginal consumers—the standard market-power distortion. Regulation reduces this distortion by dampening shock pass-through, but introduces a new inefficiency: prices depend on back books, distorting allocation away from firms with large back books.

In the equilibrium of the unregulated market, prices, market shares, and profits are i.i.d. over time. Under regulation, Proposition 2 implies that the market shares evolve as a stable

AR(1), and hence market shares and prices admit a unique stationary distribution. In what follows, we compare expected total surplus and profits in this stationary distribution. Let $\bar{W}^U(\delta,\lambda)$ and $\bar{W}^R(\delta,\lambda)$ denote the expected per-period total surplus in the MPE of the game without regulation and in the stationary distribution of the affine-strategy MPE under regulation, respectively.

Proposition 4. *$\bar{W}^U(\delta,\lambda) > \bar{W}^R(\delta,\lambda)$ for all $\delta < 0.75$. For $\delta \geq 0.75$, there exists a strictly increasing cutoff $\lambda^*(\delta)$ such that*

$$\bar{W}^U(\delta,\lambda) \gtrless \bar{W}^R(\delta,\lambda) \iff \lambda \gtrless \lambda^*(\delta),$$

with $\lambda^*(0.75) = 0$ and $\lambda^*(1) \approx 0.31$.

Regulation helps if firms are sufficiently patient and most customers are active. However, regulation always reduces total surplus if at least 31% of customers are passive. If the discount rate is 0.9, regulation hurts if at least 21% of customers are passive. And for discount rates below 0.75, regulation always reduces long-run expected total surplus.[13]

The effect of price walking regulation on each firm's profits is ambiguous: it raises new-policy prices but lowers renewal prices. Without regulation, firms extract high surplus from passive consumers, leading to intense competition for new customers via cheap policies. Regulation reduces passive-consumer surplus extraction, weakening competition for new customers and raising prices.

Let $\bar{\pi}^U(\delta,\lambda,K)$ and $\bar{\pi}^R(\delta,\lambda)$ denote each firm's expected per-period profits without regulation and in the stationary distribution with regulation.

Proposition 5. *For (δ,λ) fixed, there exists $\bar{K}(\delta,\lambda) > 0$ such that*

$$\bar{\pi}^U(\delta,\lambda,K) \gtrless \bar{\pi}^R(\delta,\lambda) \quad \iff \quad K \gtrless \bar{K}(\delta,\lambda).$$

For (K,λ) fixed, there exists $\bar{\delta}(K,\lambda) < 1$ such that

$$\bar{\pi}^R(\delta,\lambda) > \bar{\pi}^U(\delta,\lambda,K) \quad \textit{for all} \quad \delta > \bar{\delta}(K,\lambda).$$

The proposition shows that when firms are allowed to charge high enough prices, surplus

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[13] The immediate effect of regulation depends on the initial state. Regulation has a positive immediate effect on total surplus if initial market shares are sufficiently balanced ($y_t \approx 0$) and has a negative effect if they are sufficiently unbalanced ($|y_t|$ large).

extraction dominates. Without regulation, profits increase linearly in the price cap K paid by all passive consumers. Since prices under regulation are independent of K, regulation reduces profits for high enough K. The second part of the proposition shows that the competition weakening effect always dominates if firms are sufficiently patient, leading to higher profits under regulation.

Since our data covers only new policies, Proposition 3 provides our most relevant prediction. Proposition 5 determines the overall price effect (including both new and renewing policies). Since we normalized costs to zero and all consumers buy a policy, average profits equal average prices in our model.

3.4 Anticipated Regulation

We now analyze the effect of announcing the policy before implementation. A regulator announces price walking regulation in period T, credibly starting at $T+1$. The announcement is unanticipated, so prices until T follow Subsection 3.1.

We characterize the MPE starting at the announcement period T. Firms can still charge different prices to joiners and renewers in period T, but know they must charge uniform prices from $T+1$ onward (including to customers who purchase in T and remain).

Proposition 6. *There exists an MPE in which firms choose strategies:*

$$\begin{aligned} p_t^0(\Delta_t, y_t) &= \begin{cases} c\big(n + a\Delta_T\big) & \text{if } t = T, \\ c\big(m + a\Delta_t + by_t\big) & \text{if } t > T, \end{cases} \\ p_t^1(\Delta_t, y_t) &= \begin{cases} c\big(n - a\Delta_T\big) & \text{if } t = T, \\ c\big(m - a\Delta_t - by_t\big) & \text{if } t > T, \end{cases} \end{aligned} \tag{14}$$

where $n := 1 - \frac{\delta b(2-\delta b^2)}{2-\lambda(1-\delta)} \in (0, m)$*, and renewal prices* $r_T^i(\Delta_t, y_t) = K$*,* $i = 0, 1$*. Moreover, this is the unique MPE in which firms use affine strategies.*

Comparing prices in the unregulated market, at the announcement, and at the implementation, we find that the average price of a new policy rises both at the announcement and at the implementation, with price increases at each date increasing in consumer inertia.

Formally, define the average price of a new policy in the unique affine MPE as

$$\bar{p}_R^t := \frac{p_t^0(y_t, \Delta_t) + p_t^1(y_t, \Delta_t)}{2} \quad \text{for} \quad t \geq T$$

which is constant across states. As in equation (13), let $\bar{p}_U$ denote the average price of a new policy without regulation.

Corollary 1. *The average price of a new policy increases both at the announcement and at the implementation:* $\bar{p}_U < \bar{p}_R^T < \bar{p}_R^{T+1}$*. Moreover, the price increases* $\bar{p}_R^T - \bar{p}_U$ *and* $\bar{p}_R^{T+1} - \bar{p}_R^T$ *are both increasing in consumer inertia* (λ)*.*

Without regulation, firms charge low initial prices to attract passive customers, then charge K at renewal. Announcement dampens these incentives: firms anticipate charging lower future renewal prices, reducing today's investment value. However, they can still charge separate prices today. At the implementation date $T + 1$, firms face dampened incentives and cannot charge separate prices, forcing them to sacrifice back-book profits to price competitively for new customers. This further raises prices.

3.5 Model Summary

We now summarize the model's key features and testable predictions.

Assumptions. We assumed consumers live for two periods to simplify the exposition. The results extend naturally to $N \geq 2$ periods: the state consists of the market shares of the $N - 1$ outstanding cohorts together with the current demand shock, and regulation continues to generate higher prices and stable oscillatory dynamics in market shares. The characterization of MPE in affine strategies is analogous to the one in Proposition 1 and market shares follow a stable $AR(N - 1)$ process as in Proposition 2.

We focused on MPE in affine strategies. Appendix E extends our results to symmetric interior MPE in polynomial strategies. We show that any such equilibrium must be either affine or quadratic. In both cases, prices are higher with price walking regulation and market shares follow a stable oscillating AR(1) process.

Our model abstracts from adverse selection, focusing instead on how firms exploit consumer inertia. Our empirical setting justifies our focus: UK motor insurance exhibits minimal adverse selection concerns due to the near-universal compliance with mandatory coverage;[14]

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[14] In markets with active extensive margins, regulation would further reduce surplus: higher joining prices

the concentration of demand in comprehensive coverage, which limits cream-skimming incentives (Azevedo and Gottlieb, 2017); the high degree of consumer inertia, which attenuates risk-based sorting (Handel, 2013); and the ability of insurers to flexibly adjust prices based on the driving history of consumers.

Economic Mechanisms. The central feature of our model is the share of customers who remain with the same firm. This share comprises two distinct types of consumers: passive customers (fraction λ) who remain with their original firm due to switching frictions, and active customers with strong preferences for the firm's product (those located close to the firm on the Hotelling line). This distinction parallels the classic decomposition between state dependence and preference heterogeneity going back at least to Heckman (1981). Both sources account for the observed persistence in consumer choices and for firm market power, although they have different implications for pricing and welfare.

Without regulation, firms "invest" in building large back books by offering low joiner prices today, then exploit passive customers tomorrow by charging the maximum renewal price K. Active customers with strong brand preferences cannot be exploited in this way, because they switch if renewal prices rise too high. Thus, the profitability of the investment strategy depends critically on the fraction of passive customers λ: a larger λ makes the back book more valuable and justifies deeper initial discounts.

Under regulation, firms can no longer charge different prices to joiners and renewers, fundamentally changing how they monetize their back books. The inherited back book now creates a trade-off: raising prices extracts more revenue from price-insensitive, passive customers but simultaneously reduces competitiveness for active customers (both new joiners and existing customers willing to switch). Thus, firms price products with large back books less competitively post-regulation, as firms tilt toward harvesting their existing passive base rather than investing in future growth.

Importantly, regulation does not eliminate search gains. Prices respond to demand shocks and back books, with market shares oscillating as high-share products become expensive next period, turning over the identity of the cheapest offer. Moreover, firms can introduce new products to reset back-books, relaxing the pricing constraint and enabling simultaneous back-book harvesting and aggressive new-business investing.

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reduce quantity.

3.5.1 Testable Predictions

Our model delivers five testable predictions that we evaluate in Section 5:

1. **Price discounts to passive consumers** (Proposition 3 and Corollary 1): Average new-policy prices increase at both announcement and implementation, with larger increases for groups with more passive consumers (λ), as the regulation reduces the value of attracting them. We test this prediction in Section 5.1, constructing proxies for inertia λ.
2. **Back-book pricing** (Proposition 1): Post-regulation, insurers price products with larger back books less competitively, as they trade off back-book revenue against competitiveness for new customers. We document this pattern in Section 5.2.
3. **Market share dynamics** (Proposition 2): Post-regulation, market shares follow a stable, oscillating process, with products gaining share in one period becoming less competitive in the next. Section 5.3 confirms this prediction.
4. **Product proliferation**: The regulation creates product proliferation incentives, most notably for firms with large back books, as new products reset the back-book constraint for new consumers. We show substantial product proliferation by high-back-book firms in Section 5.4.
5. **Persistent gains to search**: Despite regulation, prices continue responding to demand shocks and back books, maintaining substantial search benefits for active consumers. Section 5.5 documents that switching gains remain large post-regulation.

4 Data

Our primary dataset comes from *Compare the Market*, the largest price comparison website for UK motor insurance.[15] We obtain a random sample of 8 million insurance enquiries from January 2019 to December 2023. For each enquiry, we observe three types of information: (i) customer characteristics (demographics, driving history, vehicle details, coverage type); (ii) the complete menu of quotes each customer receives, including prices, product identifiers, and insurers; and (iii) product choices for customers who click through to purchase.[16]

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[15] `https://www.actuarialpost.co.uk/article/compare-the-market-top-pcw-despite-most-expensive-quotes-22034.htm`

[16] Customers who purchase through other channels (e.g., phone) do not appear as sales in our data. Appendix Table B.1 compares our sample demographics to the UK driver population.

This granular data structure allows us to test our model's predictions. We observe menus offered to individual customers, enabling us to absorb customer characteristics with fixed effects and compare prices that different firms offer the same customer. We observe the same products offered to different customers, allowing within-product analysis of how firms vary prices across customer types while holding contract features constant. Aggregating consumer choice data through time, we construct proxies of insurers' back books. Finally, the 2022 regulatory shift provides variation across regulatory regimes.

We supplement these data with *Consumer Intelligence* survey data from individuals who renewed car insurance during July 2019 to December 2023. Respondents report their renewal price changes, whether they shopped around, potential savings from switching, and whether they switched insurers. Appendix B describes these data in more detail and characterizes customers' search and switching decisions as a function of price changes at renewal and customer characteristics.

4.1 Summary Statistics

Table 1 summarizes customer characteristics and offered menus. The median customer is 29 years old, has held a driving license for 7 years, and insures a car worth £4,855. Insurance prices vary substantially with these characteristics: young drivers face high prices that decline sharply with experience, and different cars command different premiums. We absorb this variation using 'car' fixed effects for each brand-model-year combination, following Argyle, Nadauld, and Palmer (2022).

Customers face large menus: the median customer receives 64 quotes from 26 insurers (10th percentile: 16 quotes; 90th percentile: 103 quotes). We treat products as firm-specific, so two firms offering similar contracts represent distinct products. Product fixed effects will absorb this product-level variation in our analysis.

Price variation across and within customers is substantial. The median customer receives a cheapest quote of £681, but the 90th percentile customer faces prices over three times larger. A regression of the price of insurance on product-date fixed effects gives an R^2 of only 12%, highlighting the large variation in prices across customers for the same insurance product. Even conditional on the car insured, price variation remains enormous. Within-customer variation is also large: the median difference between the 5th cheapest and cheapest offers exceeds £100, with some customers facing dispersion an order of magnitude larger. A regression of the price of insurance on enquiry fixed effects gives an R^2

Table 1: Summary Statistics

	Mean	Median	Std. dev.	10th pctile	90th pctile
Customer characteristics					
Driver age	35	29	16	19	58
Years with licence	11	7	10	0	25
Car value (£)	8459	4855	14319	900	19900
Annual mileage	7508	6000	6950	3000	10000
Offered menu					
Numb. products	62	64	33	16	103
Numb. firms	25	26	13	5	41
Offered prices and dispersion within menu (£)					
Cheapest quote	1266	681	2818	242	2445
2nd − cheapest	246	33	1584	1	416
3rd − cheapest	421	72	2228	9	722
4th − cheapest	536	101	2554	16	922
5th − cheapest	623	127	2890	23	1106
Worst − cheapest	6957	4006	17051	1075	15633
Offered prices and dispersion within menu (% car value)					
Cheapest quote	55	15	186	3	126
2nd − cheapest	11	1	121	0	15
3rd − cheapest	19	2	167	0	28
4th − cheapest	24	2	190	0	38
5th − cheapest	29	3	206	0	47
Worst − cheapest	302	87	814	12	738

Notes: This table reports summary statistics on customer characteristics and offered menus. Prices as a % of car value equal the annual premium divided by the insured car's value.

of 37%, indicating that a large proportion of variation in prices is within-menu variation. This variation stems from firms' differing cost assessments, product feature differences, and customer-insurer relationships.

Table 2 summarizes choice and search patterns. 11% of enquiries result in purchases, which we aggregate to construct back book proxies (described below). Customers overwhelmingly choose cheap options: fewer than 20% select outside the five cheapest products. Lower-ranked choices reflect non-price product characteristics, such as brand reputation, claims service quality, or add-ons (Table A.2 shows products with more features cost more).

Approximately two-thirds of customers search in multiple years. We use prior-year search behavior to proxy search propensity in Section 5. About half of customers who search in a

Table 2: Summary Statistics: Search and Choice

	Percent of sample
Selected a product	11
Customer chose:	
cheapest product	43
top 2 products	62
top 3 products	72
top 4 products	78
top 5 products	83
Customer searched:	
in one year only	36
in multiple years	64
once in year	47
multiple times in year	53

Notes: This table summarizes choice and search behavior in our sample.

given year search multiple times, potentially "shopping across terms" by adjusting enquiries to find better prices.

Motor insurance costs increased substantially over our sample period. Figure 1 shows the distribution of the cheapest quotes in 2019 and 2023. Median prices rose by 30% between these years, partly reflecting post-Covid inflation and supply-chain disruptions that increased insurers' costs.[17] Hence, our identification strategy relies on granular cross-sectional variation rather than these aggregate time-series trends.

4.2 Insurer Back Books

The 2022 regulation links prices firms charge new customers to prices charged to their existing back books. We construct two back book proxies using customer choice data.

Our first proxy is granular, closely following the regulation's structure. Price-walking rules operate at the firm-product-customer type level: firms cannot offer the same product at a higher price to existing customers than they charge to prospective customers who are otherwise equivalent. As recounted in Section 2, the regulations do not prescriptively define "equivalence"; rather, compliance with the regulations implies some convergence between the prices offered to a prospective customer and existing customers who are similar along other dimensions. We define similarity based on three key inputs into underwriting models: the

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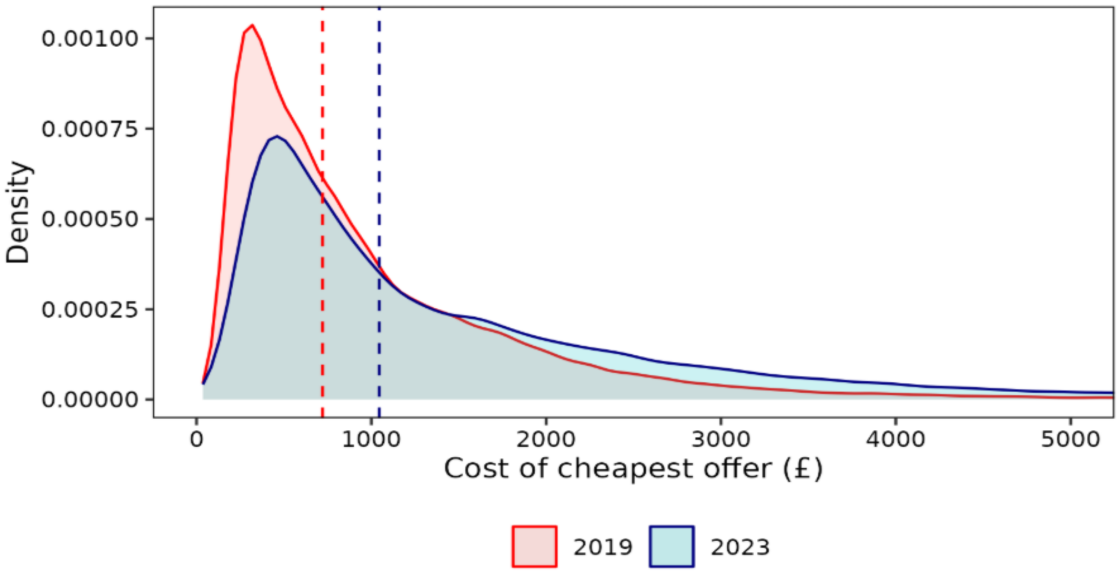
[17] `https://www.ft.com/content/04a7ba0b-9ed1-4191-819c-9f88faa20a34`

Figure 1: Price of Insurance, 2019 and 2023



Notes: This figure shows cheapest-quote distributions in 2019 and 2023. Dashed lines show medians.

age of the driver, the value of the collateral, and the level of coverage offered; our discussions with market participants and policymakers confirm that these definitions are reasonable. In particular, we define customer types based on (a) the driver's birth year; (b) car value quintile; and (c) coverage level.

At each time t, we measure customers of a given type likely to renew a specific product using choice data from one and two years prior. For example, consider a customer born in 1980 with a car valued in the bottom quintile and whose policy expires on September 5, 2023. We compute the firm's back book for this product by counting customers of this type who purchased the same product in the 30 days around September 5, 2022 and September 5, 2021. Unless they switched, these customers renew around September 5, 2023, when the firm attracts new customers of the same type. Regulation ties new-customer prices to renewal prices for these customers. We convert counts to shares by dividing by the sum across all firms and products within each type.

Our second proxy aggregates to the firm level. Product launch/retirement decisions occur at the firm level, requiring a broader measure. For firm f at time t, we count all customers who purchased from the firm before t, dividing by total contracts across all firms over the

Table 3: Back Books: Summary Statistics

	Mean	Median	90th pctile	99th pctile
Product-segment back book				
Count	1	0	2	71
Share	0.006	0.000	0.018	1.000
Aggregate firm back book				
Count	9052	2365	28681	68752
Share	0.017	0.004	0.054	0.129

Notes: This table summarizes proxies of insurer back books defined in Section 4.2. Segment-specific back books cover the whole sample; aggregate back books are measured as of December 2021.

same period to obtain a share.

Table 3 summarizes both measures. The first two rows show product-level back books. Fine-grained product and customer type definitions combined with two-month measurement windows yield small average back books, but substantial cross-product and cross-segment variation. Section 5.2 exploits this variation to study how firms condition pricing on product back books. The third and fourth rows show aggregate firm back books. The average insurer attracted thousands of customers in our sample, but this distribution is highly skewed: small or new firms have minimal back books, whereas the largest captured over 10% of customers.

5 Empirical Results

The model in Section 3 predicts a set of testable implications about the transition dynamics in the market around the announcement and introduction of the price walking regulation in a range of outcome variables, placing an important role on insurers' back books as well as consumer characteristics in explaining cross-sectional variation in these outcomes. In this section, we uncover five main patterns in the *Compare the Market* data that illustrate how the 2022 pricing regulations have affected the motor insurance market. Overall, we find that these empirical results line up well with the model's main predictions.

5.1 Price Discounts to Passive Consumers

We begin our analysis by studying patterns in the initial price discounts offered to customers around the 2022 pricing regulations. The model in Section 3 predicts that these discounts are increasing in the likely level of inertia of customers, and thus customers more likely to be inert should experience larger price increases due to the regulation. In this section, we show that this is corroborated in the data: Customers who are ex ante more likely to be inactive experience steeper pre-2022 increases in the cheapest available quote, with no additional relative change once the rule comes into force.

Testing this hypothesis requires us to identify consumers who are likely to be passive. To do so, we leverage our data on repeat searches by customers. We restrict the sample to those customers in our sample of searchers who selected a product to purchase. We then estimate probit regressions to predict the likelihood that these same customers *do not* search again twelve months later (i.e., behave as passive rather than active customers) as a function of the observed customer characteristics.

Table 4 shows the marginal effects from these regressions. Conditional on having searched once, the likelihood of searching again is lower for customers who have cheaper cars, shop for non-comprehensive insurance (presumably in an effort to keep costs down), are out of work due to sickness or disability, and are young.[18] We use the estimates from Table 4 to compute a predicted level of inertia for every enquiry in our sample.

We should point out that the conditioning event of having searched and purchased in year 0 may not be equally selective across customer types. For young drivers taking out their first policy, year-0 search is likely non-discretionary, so the sample of year-0 searchers is close to the full population of young drivers. For established drivers, by contrast, year-0 search is a choice, and those who make it are positively selected on being active types: the most passive established drivers auto-renew without visiting a price comparison website and never appear in our estimation sample. Accordingly, our probit estimates in Table 4 need not match the relationship between inertia and age in the population. In Appendix C, we perform several robustness checks on our measurement of inertia, and show that our findings are robust along various dimensions. We therefore interpret our inertia index as a valid proxy for revealed future search propensity, correlated with the model's λ parameter.

We regress the cheapest offer on the menu for each enquiry on this inertia proxy, allowing the coefficient on inertia to vary through time. Given that our measure of inertia correlates

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[18] This finding may appear at odds with the fact that the customers actively searching for car insurance tend to be younger, as Table 1 reports. The reconciliation lies in the distinction between unconditional and conditional search probabilities. Unconditionally, young drivers search more frequently, as they enter the market in large numbers and have no existing policy to renew. Conditionally on having searched and purchased, however, they are less likely to search again, because the older drivers who also searched in year 0 are positively selected on being active types. Table B.2 in Appendix B uses the *Consumer Intelligence* dataset to confirm that young drivers search more frequently in the unconditional sense.

Table 4: Inertia and Customer Characteristics

Dependent variable:	Inert next year (1)
Age <20	0.353***
	(0.005)
Age 20-29	0.223***
	(0.003)
Age 30-39	0.198***
	(0.003)
Age 40-49	0.170***
	(0.003)
Age 50-59	0.136***
	(0.003)
Age 60-69	0.090***
	(0.004)
Insuring cheap car	0.057***
	(0.002)
Disabled	0.066***
	(0.010)
Third-party, fire & theft cover	0.045***
	(0.005)
Third-party cover	0.036***
	(0.009)
Unemployed	-0.035***
	(0.006)
Observations	592,734
Pseudo R-squared	0.019
Mean dependent variable	0.38
Std. dev. dependent variable	0.49

Notes: This table shows how the likelihood that a customer does not search a second time (conditional on having searched and selected a product) varies with demographics. The reported estimates are average marginal effects from probit regressions of a dummy for whether a customer searched twelve months after taking out insurance, for customers taking out insurance at least one year before the end of our sample. A car is cheap if it is in the bottom 20% of car values. 'Disabled' denotes a driver who is out of work due to illness or disability. The omitted age category is over-70. The omitted coverage category is comprehensive insurance.

with other customer characteristics that affect risk (for example, younger drivers tend to be riskier as well as more inert), we refrain from interpreting the level of its correlation with prices. Hence, we focus our test on the *time-varying* relationship between prices and the index. Table 5 reports the coefficient estimates of this regression, for differing control variables, with the reference period being 2022, the first year in which the policy is in place. Across each specification, the coefficient on the inertia proxy increases each period until 2021, when the policy was finalized, and changes little thereafter. In Figure 2 we plot the results of a monthly version of the third regression, with January 2022 as the reference period. There is a linear increase in the coefficient of inertia in the years leading up to the policy, and little change afterwards.

This striking pattern—a relative increase in prices in the run-up to the policy with no further change thereafter—suggests that the regulation reduces the desirability of inactive customers. The selection concern described above does not threaten this inference: The mechanism whereby passive established drivers are absent from the year-0 search sample is approximately time-invariant, and cannot generate the smooth pre-trend followed by stabilisation precisely at the regulation date that Figure 2 documents. The model predicts that the regulation should affect prices on announcement as well as on implementation. As explained in Section 2, the thrust of the policy could be anticipated years in advance of its implementation. Three years before the policy, in 2019, inactive customers could still be price-walked for a period of time without any cost in terms of competitiveness for new business. As the year of implementation approaches and the nature and timing of the regulation becomes clear,[19] the benefit of attracting new inactive customers shrinks, as there are fewer years in which they can be "costlessly" price walked. Post-policy, there is no further change to how desirable inactive customers are relative to searchers.

5.2 Back-book Pricing

We next study how the regulation affected insurers' pricing of their available products as a function of their back books. The model predicts that once price walking is banned, products with large back books should be less competitively priced. In this section we combine our empirical measure of insurer back books with pricing behaviour around the introduction of the regulation to test this prediction.

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[19] The regulatory process followed a standard and predictable process from its inception in 2018, with the price walking ban being floated as an option in 2019, consulted on in 2020, finalized in 2021, and implemented in 2022 (Financial Conduct Authority, 2021).

Table 5: Pricing and Customer Inertia

	(1)	(2)	(3)
Inertia	5.605***	3.457***	3.724***
	(0.076)	(0.052)	(0.057)
Inertia × {Year = 2019}	-0.421***	-0.733***	-0.615***
	(0.086)	(0.054)	(0.063)
Inertia × {Year = 2020}	-0.267***	-0.387***	-0.345***
	(0.079)	(0.054)	(0.060)
Inertia × {Year = 2021}	-0.089	0.063	0.094
	(0.081)	(0.057)	(0.063)
Inertia × {Year = 2023}	0.007	-0.026	0.039
	(0.074)	(0.051)	(0.054)
Fixed effects			
Month	Yes	Yes	Yes
Product	No	Yes	Yes
Car	No	No	Yes
Observations	7,282,117	7,282,113	7,265,715
R-squared	0.29	0.54	0.63
Mean dep. var.	6.61	6.61	6.61
Std. dev. dep. var.	0.92	0.92	0.92

Notes: The table shows the relationship between the price of insurance and the enquirer's inertia proxy over time. We estimate the inertia proxy using the fitted values from the probit regressions summarized in Table 4. We then regress the log of the cheapest offer of insurance for enquiry i on this inertia proxy, allowing the slope coefficient to vary by year. Car fixed effects are indicator variables for a given combination of vehicle model and year of production. Bootstrap standard errors are clustered at the level of the enquirer. Signif. Codes: ***: 0.01, **: 0.05, *: 0.1.

Firms can adjust their quoting behaviour for a particular product and a particular customer in two ways: they can change the price at which they offer a product to the customer, or they can choose not to offer that particular customer the product at all. If firms with large back books are unable to compete for new customers, they may quote them a high price, or may simply not offer them a quote. To capture the latter behaviour, we must define the set of potential products that could be offered to a customer. To do so, we create an expanded menu for each customer containing all products that were offered to any customer in the month of the enquiry, regardless of whether it was offered to that specific customer.[20]

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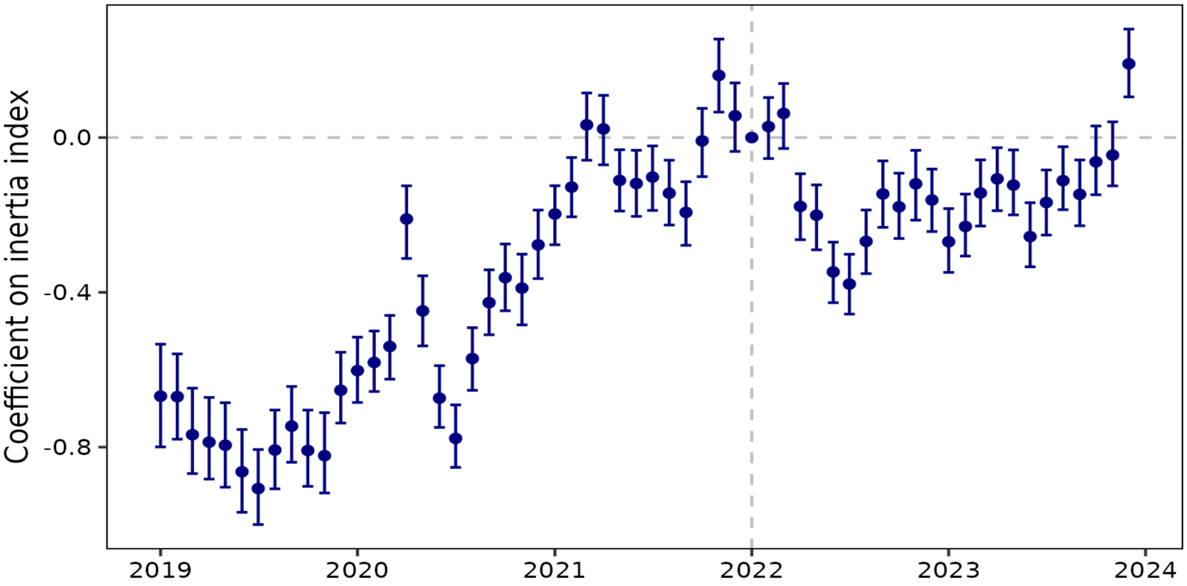
[20] Clearly, for products not offered to the specific customer, we do not see what the price would have been, had it been offered.

Figure 2: Pricing and Customer Inertia Through Time



Notes: The figure displays, through time, the relationship between the price of insurance and the enquirer's predicted level of inertia. We estimate a version of the third column of Table 5 where we allow the coefficient on inertia to vary by months. The reference month is January 2022. Error bars show 95% confidence intervals based on bootstrapping, where standard errors are clustered at the level of the enquirer.

We relate how competitive a firm's quotes to customers are to the firm's back book of customers of the same type (i.e., using the different proxies of the back book described earlier). We measure competitiveness based on whether a firm was at the top of the menu a customer faces (recall that Table 2 shows that top-of-menu products have a 43% likelihood of being chosen).

More precisely, we run different specifications of the following difference-in-difference linear-probability regression:

$$\text{top}_{ijt} = \beta_0 \text{BB}_{\theta(i)jt} + \beta_1 \text{Post}_t \text{BB}_{\theta(i)jt} + X_{ijt}\Gamma + \epsilon_{ijt}, \tag{15}$$

where the dependent variable top_{ijt} is an indicator variable equal to 1 if product j gives the cheapest quote in response to customer i's enquiry at time t, and 0 otherwise; $\text{BB}_{\theta(i)jt}$ denotes product j's back book for the type $\theta(i)$ of customer i at time t; Post_t is an indicator variable equal to 1 after the implementation of the pricing regulation in January 2022, and 0 otherwise; X_{ijt} is a vector of controls, including time fixed effects; and ϵ_{ijt} is an unobservable.

The different specifications allow for more or less aggregation of our dependent variable, for example whether j is the cheapest product in a customer's full menu, or if it is just the cheapest offered by firm f in that menu. Moreover, because we observe menus for each individual enquiry, we can perform our analysis within an individual customer enquiry. That is, our most stringent regressions compare the quotes that different insurers offer to *the same individual*. Additionally, we allow the coefficient on the back book to vary across firms—i.e., we look within individual firms to evaluate how much the effect of having a large back book on quoting behaviour changes following the implementation of the policy.[21]

Table 6 reports the coefficient estimates of these different specifications of regression (15). Columns (1)–(3) report the results for the specifications with the dependent variable defined across all firms—i.e., top_{ijt} equals 1 if product j is the cheapest quote in response to customer i's enquiry at time t, and 0 otherwise. Specifications (1) and (2) are similar, although (2) is more stringent as it includes enquiry fixed effects. Specification (3) features firm-level slopes. Across all specifications, the key difference-in-difference coefficient of interest (i.e., the effect of the back book after the policy relative to before the policy) is negative, meaning that firms with large back books price less competitively after the policy change.

The results are economically large. According to the estimates in columns (3), moving from the 1st to the 99th percentile of the back book (Table 3) implies a decrease in the probability of a product giving the cheapest quote of 9 percentage points. This is over ten times the average unconditional probability that a given product is the cheapest quote.

The left panel of Figure 3 shows results of a time-varying version of the regression in Column (3) in Table 6, with quarterly coefficients on the back book, the main variable of interest. The effect on pricing kicks in sharply at the time the policy is introduced. There are no significant pre-trends. This is intuitive: in Section 5.1 we showed that firms reduced their teaser rates for inert customers in anticipation of the policy's implementation, as the expected present value of attracting an inert customer decreased. However, the ability of firms with large back books to price competitively for new business only fell when the policy was actually implemented, and firms had no incentive to act prior to this implementation.

Columns (4)–(6) of Table 6 report the estimates of the specifications in which the dependent variable is more disaggregated—i.e., top_{ijt} equals 1 if product j is the cheapest amongst all products offered by firm f in response to customer i's enquiry at time t, and 0 otherwise. In each specification, after the policy change, products with large back books become less

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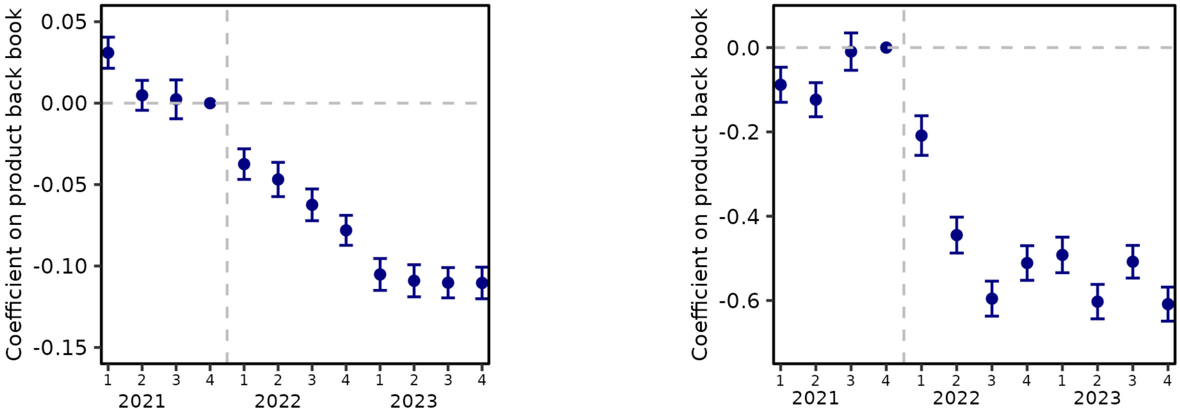
[21] Anecdotally, a minority of firms did not engage in price walking even before the policy.

Table 6: Competitiveness and Back Books

	Cheapest in menu			Cheapest in firm		
	(1)	(2)	(3)	(4)	(5)	(6)
Product back book	0.065***	0.065***		0.293***	0.318***	
	(0.00182)	(0.00182)		(0.00986)	(0.00721)	
Product back book × post	-0.091***	-0.091***	-0.090***	-0.388***	-0.423***	-0.451***
	(0.00232)	(0.00232)	(0.00257)	(0.01085)	(0.00849)	(0.00886)
Fixed effects						
Segment-month	Yes	No	No	Yes	No	No
Segment-product	Yes	Yes	Yes	Yes	Yes	Yes
Product-month	Yes	Yes	Yes	Yes	Yes	Yes
Enquiry	No	Yes	Yes	No	No	No
Firm-enquiry	No	No	No	No	Yes	Yes
Varying slopes						
Firm back book	No	No	Yes	No	No	Yes
Mean dep. var.	0.007	0.007	0.007	0.184	0.184	0.184
Observations	729,584,529	729,584,529	729,584,529	729,584,529	729,584,529	729,584,529

Notes: This table reports the coefficient estimates of different specifications of regression (15). Standard errors are clustered at the product-segment-month level. Signif. Codes: ***: 0.01, **: 0.05, *: 0.1.

Figure 3: Pricing and Back Books



(a) Across-Firm Pricing & Back Books

(b) Within-Firm Pricing & Back Books

Notes: The figures show, through time, the relationship between how competitively a product is quoted and its back book. The left panel displays the coefficients of the regression $\text{top}_{ijt} = \beta_t \text{BB}_{\theta(i)jt} + X_{ijt}\Gamma + \epsilon_{ijt}$, where all variables are defined in the text and the controls are those in the third column of Table 6. The right panel shows results of the regression $\text{top}_{ijt} = \beta_t \text{BB}_{\theta(i)jt} + X_{ijt}\Gamma + \epsilon_{ijt}$, where variables are as defined in Section 5.2 and controls are those in the sixth column of Table 6.

likely to be the cheapest product offered by a firm. The magnitudes are again economically large, though slightly smaller than the across-firm results of Columns (1)–(3). Moving from the 1^{st} to the 99^{th} percentile of the back book (Table 3) implies a decrease in the probability that a product is a firm's cheapest offer of 45 percentage points. This is around 2.5 times the average probability that a given product is the cheapest a firm offers.

The right panel of Figure 3 shows results of a time-varying version of the most stringent product-level regression (Column (6) in Table 6). Once again, the effect coincides with the introduction of the policy, and there are no significant pre-trends.

These results confirm the model's predictions regarding pricing and back books in Proposition 1. Products that have attracted a large amount of a particular type of customer in the past are now less competitive when competing for new customers of this type. This operates across all products: products with large back books are less likely to appear at the top of a customer's menu than products with no back book after the regulation. It also operates within firms: firms adjust prices across the products they offer in response to each product's back book, such that if a given product becomes uncompetitive by virtue of having a large back book, the firm can steer consumers to alternative products by pricing them competitively.

5.3 Market Share Dynamics

We now test the model's prediction that the policy introduces negative autocorrelation in insurance product market shares. To test this in the data, we first define a product's market share at the month-segment level. That is, for each product j quoted to customer type θ (as defined in Section 4.2) in month t, we compute its market share $s_{j\theta t}$ as the number of times product j was selected by that type, as a share of all product selections by that type.

We then relate these market shares $s_{j\theta t}$ to market shares 12 months earlier $s_{j\theta t-12}$, using the following autoregressive specification:

$$s_{j\theta t} = \rho_t s_{j\theta t-12} + \delta_{j\theta} + \epsilon_{j\theta t}, \tag{16}$$

where ρ_t is the autoregression coefficient, which we allow to vary across time, $\delta_{j\theta}$ are product-type fixed effects, and $\epsilon_{j\theta t}$ is an unobservable. We plot the results of this regression in Figure 4. The points show monthly estimates of ρ_t, whilst the horizontal lines show their average level at the beginning of our sample period and after the introduction of the policy.

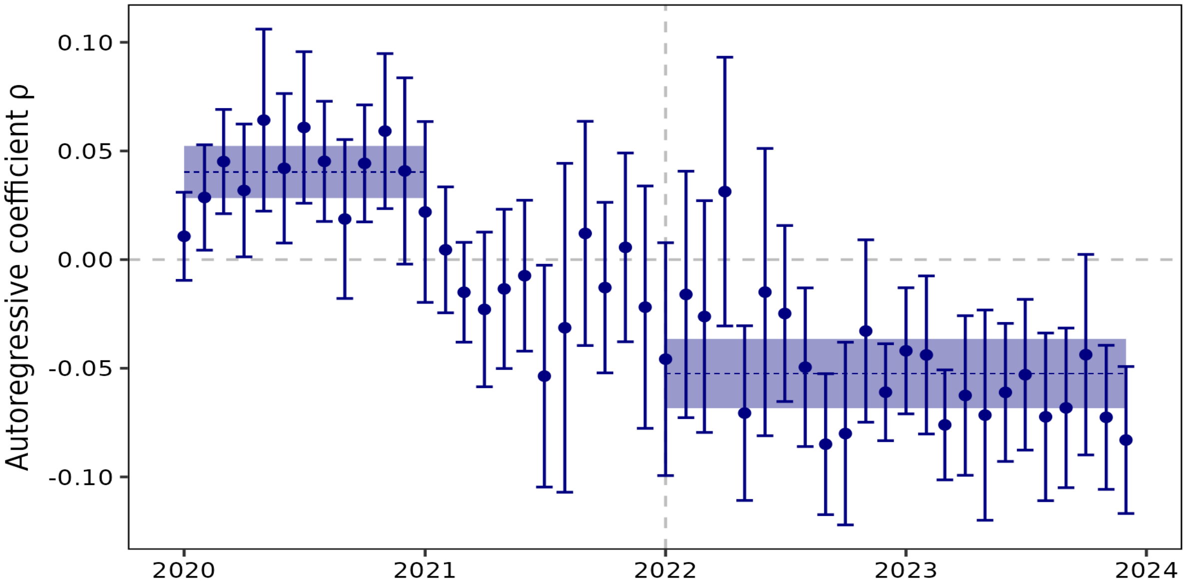
The patterns are striking. Initially market shares exhibit positive autocorrelation, namely products that are popular in a given segment in one year are more likely to be popular the following year. As the policy is anticipated and then implemented, the autocorrelation reverses sign, such that being in greater demand one year is associated with being less popular the following year.

These results confirm the prediction of Proposition 2 regarding the impact of pricing regulation on competition. As we found in Section 5.2, after the policy the presence of a large back book inhibits a product's competitiveness. As shown in Figure 4, this pattern in pricing causes the market shares of products to oscillate through time after the policy.

5.4 Product Proliferation

We now study the introduction of new products around the change in regulation. As explained in Section 3.5, the regulation introduces an incentive for firms to introduce new products, and this incentive is stronger for firms with large back books. We find support for this prediction in the data: following the change in regulation in 2022, the number of

Figure 4: Oscillating Market Shares



Notes: This figure shows the results of the autoregressive specification in Equation 16. Points show the estimated autoregressive coefficients ρ_t for each month, with error bars showing 95% confidence intervals. The dashed horizontal lines show the estimates when this parameter is allowed to vary only before vs. after the policy's implementation, with the ribbons showing 95% confidence intervals. Standard errors are clustered by type θ and product j.

products offered by insurers increased sharply, and more so for firms with large back books.

As a warm-up to more tightly identified evidence, the left panel of Figure 5 displays some interesting aggregate effects. The black dashed line shows that the average firm had 25% more products on offer at the end of our sample period than it did at the beginning.

We now disaggregate this pattern of product proliferation across firms. The left panel of Figure 5 displays the number of products offered per firm for groups of firms with back books of different sizes. We focus on firm-level (rather than segment-specific or product-specific) back books because, once a product is introduced, it is generally introduced to many different segments.[22] We focus on the back book over the whole prior sample period, rather than just those contracts renewing in a given month, as once a firm introduces a product it tends to offer this product for a long period of time.[23]

We find that firms with large back books fully account for the increase in the total number

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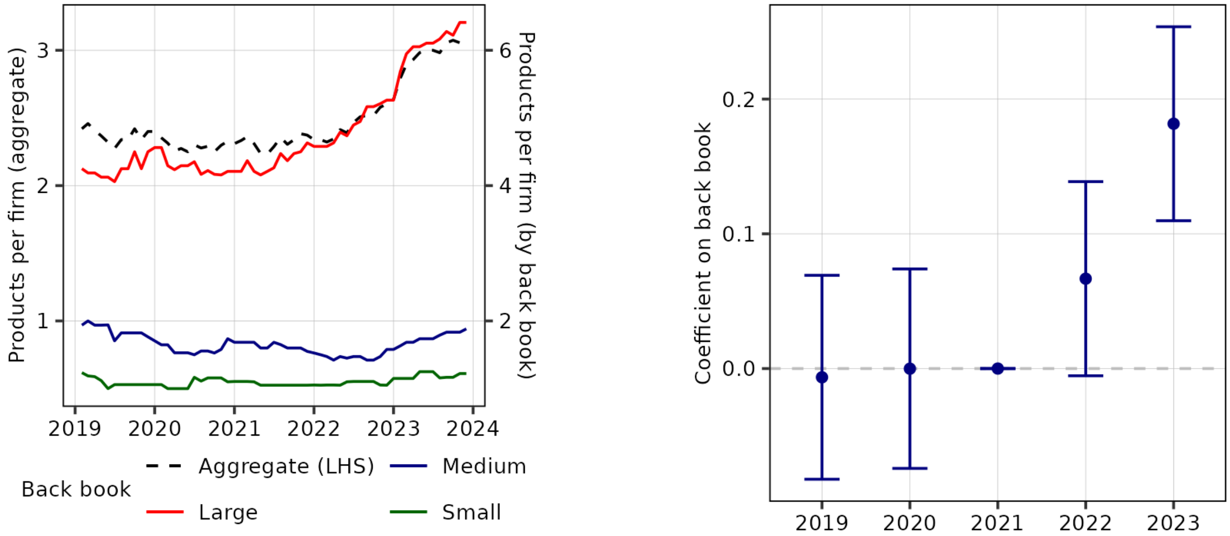
[22] On average, a product returns a quote in 43% of enquiries.

[23] For example, of the products quoted in 2019, two-thirds were still being quoted in 2023.

Figure 5: Products per Insurer, 2019–2023



(a) Trends by Back Book **(b) Difference-in-Difference**

Notes: These figures show the number of products offered by insurers through time and by the size of the firm's back book. The black dashed line in the left panel shows the aggregate number of products offered each month by all insurance firms, divided by the total number of firms. The colored lines in the left panel show the average number of products per insurance firms according to the size of the firm's back book. Back books are defined as in Section 4.2 and are grouped into tertiles. The right panel shows the coefficients on a firm's back book in the two-way fixed effects regression relating a firm's products to its back book shown in equation 17. The points show the coefficients on the back book each year, with 95% confidence intervals.

of products on offer after the policy. The largest (top-tercile) firms with the largest back books increased the number of products they offered by approximately 50% after the policy, from approximately 4 to 6. All other firms left their product offerings broadly unchanged.

We formalize this result in a difference-in-difference regression. We relate the total number of products a firm offers in a given month prods_{ft} to its back book BB_{ft}, using the following regression:

$$\text{prods}_{ft} = \beta_t \text{BB}_{ft} + \delta_t + \delta_f + \epsilon_{ft}, \tag{17}$$

where δ_t and δ_f are year and firm fixed effects, respectively. As explained in Section 4.2, a firm's aggregate back book share in a given month is defined as all customers who shopped with that firm in all prior months, as a percentage of the total customers across all firms in the same period.

The right panel of Figure 5 displays the estimates of the coefficients β_t. These estimates formalise the intuition from the time series evidence. There is no evidence of any pre-trends. After the policy implementation, we observe a sharp increase in the size of menus offered by the firms with the most incentive to expand their menus, that is those with large back books.[24]

5.5 Persistent Gains to Search

The 2022 pricing regulations aimed to eliminate direct price discrimination between passive and active consumers. The model predicts that these regulations will not eliminate the gains to search, as prices will still increase for inactive customers. This section confirms this prediction, providing evidence that the regulation did not eliminate customers' benefits from searching for and switching across insurers, which remained large.

To show this, we construct a simple proxy measure of the benefits from switching insurance products. We leverage the fact that we observe a subset of customers who purchase an insurance product one year, and then search again in the following year. When they subsequently search, we compare the price of their original product one year on—to which they would automatically renew—to the price they could get if they switched products instead.

More precisely, we restrict our sample to customers who purchase a product in a given year and search again the following year, and for whom the product chosen in the first year appears in the menu in the second year. We then compute the savings they would get if they switched to the cheapest product in the menu rather than renewing the insurance policy they chose initially.[25,26]

The left panel of Figure 6 shows the distribution of these savings across customers, as a percentage of the price of the cheapest option in a customer's menu. It shows that the median customer would pay 40% more by renewing their existing product rather than switching to

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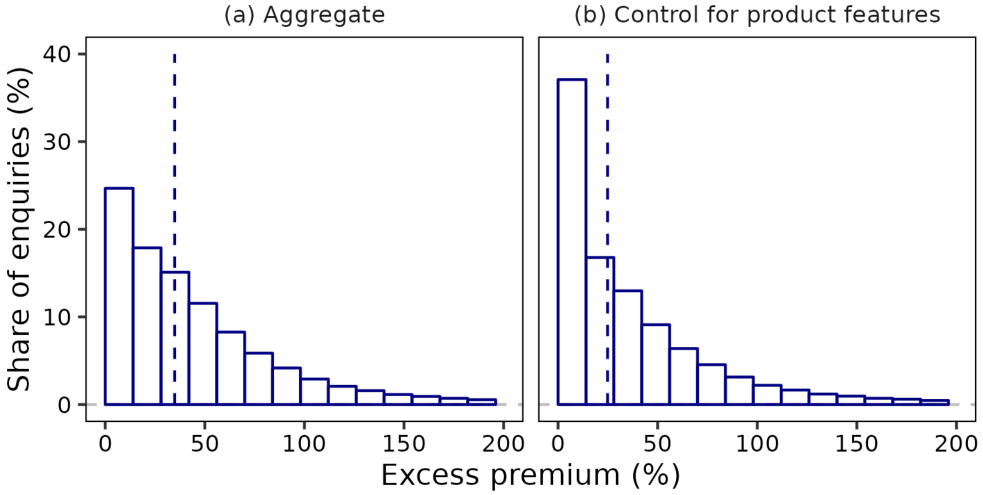
[24] The magnitudes of the coefficients in the right panel of Figure 5 are smaller relative to the magnitudes in the left panel, because the largest firms fully account for the increase in product offerings (i.e., the effect is non-linear) whereas equation (17) is linear in the back book.

[25] This approach assumes that the quote a customer receives from their existing insurer when renewing their policy is the same if they search online as if they passively auto-renew. This need not be the case pre-policy, as insurers can offer a lower price when a customer searches than their auto-renewal price, and indeed have an economic incentive to do so in order to retain them. The policy, however, requires that auto-renewing customers cannot be discriminated against relative to otherwise equivalent customers shopping via the same channel by which the customer initially shopped. Thus, the price we observe in our data cannot be below their auto-renewal price, whilst insurers typically have little incentive to set it above the auto-renewal price. Hence, we show these gains to search only for the period after the policy's implementation.

[26] Naturally, this is not a representative subsample. We expect the gains to search for the population as a whole, who are on average more inert than these repeat-searchers, to be larger than for this subsample.

Figure 6: Costs of Inertia



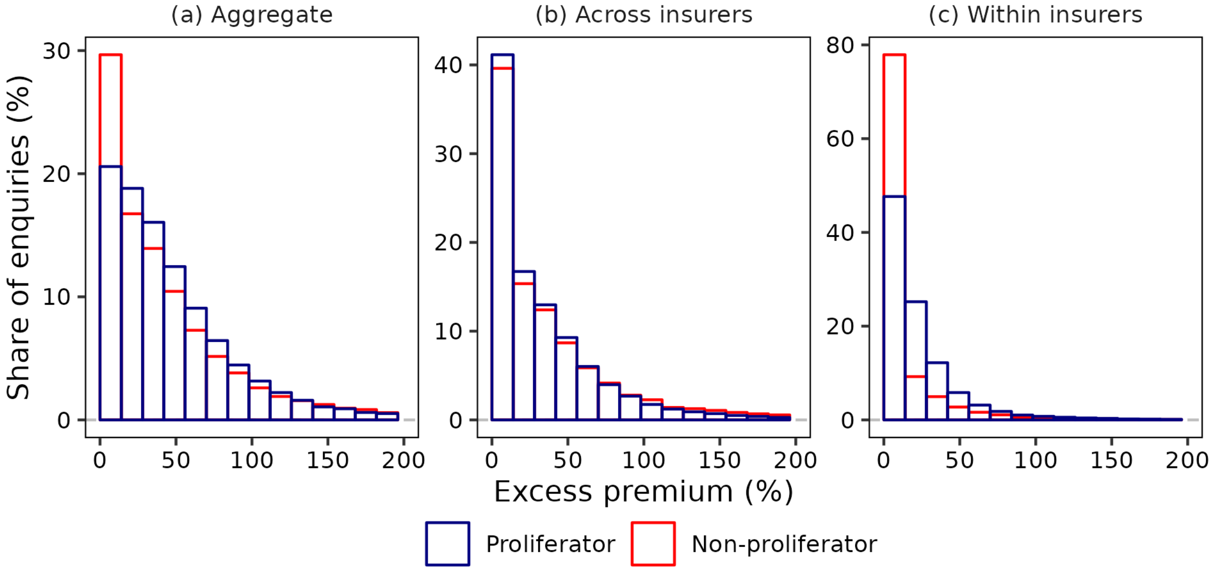
Notes: This figure shows the distribution of the excess cost of a customer's chosen product one year after they chose it. We take the subset of customers that purchase a product and search again the following year. We define the incumbent product as the product purchased in the first year. The excess cost equals the extra percentage cost of the incumbent product in year two relative to the cheapest offer in the customer's menu in year two. The left panel shows the distribution of total excess costs. The right panel repeats this calculation, then subtracts the excess cost that the customer paid the previous year, and takes the maximum of this number and zero. Dashed vertical lines show medians.

the cheapest available online quote.

The right panel in Figure 6 controls for differences in the non-price features of different products. Customers may pick a product that is not the cheapest if it has some desirable non-price characteristics. We account for this in Figure 6 by subtracting the initial difference in price between the product chosen and the cheapest product available at that time. Thus, if a customer chose a product 20% more expensive than the cheapest in one year, and a year later the product remains 20% more expensive than the cheapest, we would compute their savings as 0. These non-price characteristics explain some of the savings from switching products, but even after controlling for them, the median customer pays over 20% more than they need to a year after taking out insurance. This 20% represents the customer's chosen product becoming more expensive relative to alternative products the customer was offered, but opted not to purchase.

Overall, these patterns are consistent with insurers continuing to profit from inert customers, even when they are prevented by the price-walking regulations from directly dis-

Figure 7: Costs of Inertia and Product Proliferation



Notes: This figure shows the distribution of the excess cost of a customer's chosen product one year after they chose it, separated according to whether their chosen insurer introduced a new product in the meantime. As in Figure 6, we take the subset of customers who purchase a product and search again the following year. We define the incumbent product as the product purchased in the first year. The excess premium equals the extra percentage cost of the incumbent product in year two relative to the cheapest offer in the customer's menu in year two. The blue bars in the first panel show the distribution of total excess costs for those customers whose insurer introduced a new product in the year since they initially shopped ('proliferators'). The red bars repeat the computation for those whose insurers did not ('non-proliferators'). The second panel repeats these calculations for across-insurer gains to search, defined as the difference between the incumbent insurer's cheapest product and the cheapest offer on the customer's menu, as a percentage of the cheapest offer. The third panel repeats these calculations for within-insurer gains to search, defined as the difference between the customer's incumbent product and their insurer's cheapest product, as a percentage of the cheapest offer on their menu.

criminating against their existing customers.

We now study heterogeneity in these gains to search across insurers. As discussed in Section 3.5, insurers can ease the trade-offs imposed on them by the regulation by introducing new products, which enables them to raise prices for their back book whilst competing for new business with new products. In the first panel of Figure 7 we show how the gains to search vary depending on whether a customer's initial insurer introduced a new product onto their menu the following year. The gains from search are significantly larger than for those insurers that did ("proliferators") than those that did not.

We further explore this difference by disaggregating a customer's gains to search into a within-insurer component and an across-insurer component. Within-insurer gains to search are defined as the difference in price between the customer's incumbent product and the cheapest product offered by the customer's incumbent insurer. Across-insurer gains to search are the difference between the cheapest product offered by the incumbent insurer, and the cheapest option on the menu overall. We express both these measures as a percentage of the cheapest offer on the menu, such that they sum to our measure of aggregate gains to search.

The second and third panels of Figure 7 show within-insurer and across-insurer gains to search, split once again according to whether the customer's insurer was a proliferator or not. The across-insurer gains are similar for proliferators and non-proliferators, consistent with the two types being equally able to compete for new business. The within-insurer gains to search are significantly larger for proliferators. Thus whilst the proliferating firm can set a minimum price as low as that of other firms, its existing customers pay significantly more than this minimum.

Together, the patterns in Figure 7 highlight the role of product proliferation in the regulatory arbitrage at play here. Proliferation is a way by which insurers can continue to compete for new business, whilst increasing the prices paid by their back book.

6 Conclusion

The central question of this paper is whether direct regulatory intervention can protect consumers in markets where inertia is widespread and suppliers can adapt their behavior. We study this question through the lens of a uniquely informative natural experiment: the 2022 UK regulation banning "price walking" in motor insurance, whereby insurers kept prices low for new customers while systematically raising them for existing ones.

We develop a theoretical framework that captures the key trade-off regulation imposes on firms: charging higher prices to exploit their existing customer base makes them less competitive for new business. The model generates five testable predictions about firm behavior. We find strong empirical support for all five using granular search, choice, and pricing data from *Compare the Market*, the largest price comparison website in the UK motor insurance market.

The findings tell a coherent and multifaceted story about how firms adapted. First, as the regulation approached, insurers reduced the price discounts they offered to customers most likely to be inert—that is, the very customers the regulation was designed to protect.

Second, once the regulation took effect, firms with large back books priced their established products less competitively, steering new customers toward alternative offerings. Third, this pricing behavior caused market shares to oscillate over time, as products that gained customers in one period became expensive in the next. Fourth, firms with the largest back books responded by substantially expanding their product offerings, increasing their menus by approximately 50%, thereby enabling them to harvest their existing customer base while remaining competitive for new business. Fifth, and most consequentially, the gains from actively searching and switching remained large throughout: post-regulation, the median customer still pays around 40% above the cheapest available price by not switching, with product proliferation emerging as one of the key mechanisms sustaining this penalty.

Taken together, these results indicate that regulating markets with inertial consumers is complex: when direct price discrimination is banned, sophisticated suppliers find indirect routes to the same outcome. Prior research has shown that softer interventions, such as encouraging competition and nudging consumers toward better choices, often fail to overcome the combination of consumer inertia and strategic firm behavior. Our findings suggest that more muscular approaches, which directly constrain how firms can price across their customer base, face similar challenges, because firms respond by restructuring their product offerings to exploit inertia. Indeed, our finding on the oscillation of market shares through time—as different firms alternately exploit their back books and price competitively to attract customers to subsequently exploit—creates a transient and elusive target for regulators following the imposition of muscular regulation. Designing regulation that is robust to these endogenous supplier responses remains an open and important problem for consumer protection policy.

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Online Appendix for "Regulating Inaction: The Case of Price Walking"

A Additional Tables and Figures

Table A.1 reports price regulations in different jurisdictions similar to UK's price-walking regulations.

Table A.2 provides an example of the products offered by a large insurer.

Figure A.1 displays the menu of products presented by a price comparison website.

Table A.1: Regulatory Actions on Pricing Practices — Global Timeline

Adoption	Jurisdiction	Competent Authority	Scope
2013	Netherlands	Netherlands Authority for the Financial Markets (AFM)	Lending
2014	South Korea	Korea Communications Commission (KCC)	Telecom
2014	United States	State insurance departments	Insurance
2019	Australia	Australian Energy Regulator (AER)	Energy
2020	New Zealand	Electricity Authority (EA)	Energy
2021	Australia	Australian Securities and Investments Commission (ASIC)	Insurance add-ons
2022	United Kingdom	Financial Conduct Authority (FCA)	Insurance
2022	Ireland	Central Bank of Ireland (CBI)	Insurance
2025	Colombia	Comisión de Regulación de Comunicaciones (CRC)	Telecom
2026	Canada–Ontario	Financial Services Regulatory Authority of Ontario (FSRA)	Insurance

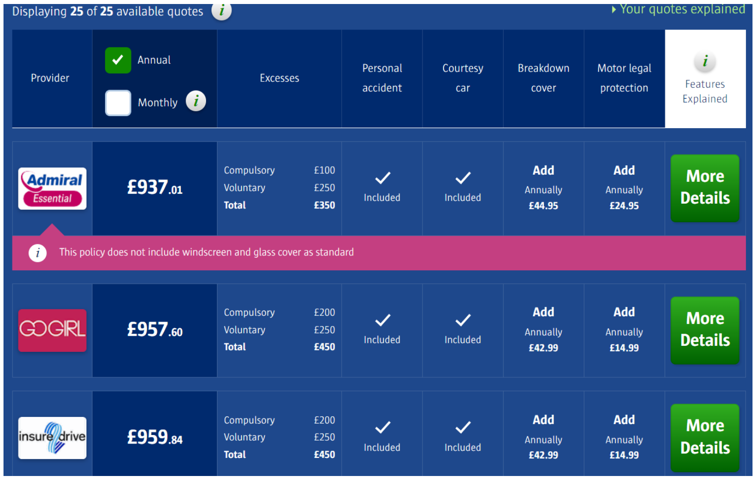
Notes: In the United States, 20 states have adopted measures on pricing practices since the first adoption by Maryland on 31-10-2014. Jurisdictions include (postal abbreviations): MD, OH, CA, NY, FL, VA, VT, WA, IN, PA, DC, ME, MT, RI, DE, CO, MN, CT, AK, MO, MI.

Table A.2: Major Insurer's Products

Feature	Platinum	Gold	Standard	Essential
Motor legal protection	✓	✓	optional	optional
Roadside breakdown cover	✓	optional	optional	optional
Personal belongings cover	£300	£300	£200	×
Windscreen cover	✓	✓	✓	×
New car replacement	✓	✓	✓	×
Driving other cars (conditional)	✓	✓	✓	×
European cover for up to 90 days	✓	✓	✓	×
Audio equipment cover (aftermarket)	✓	✓	✓	×
Onward travel	✓	✓	×	×
Cost for indicative quote	£884.69	£840.55	£816.04	£750.76

Notes: This table shows the four products offered by a major UK car insurer. Products are vertically differentiated, with a higher price charged for policies with extra features.

Figure A.1: Example of Price Comparison Website Menu



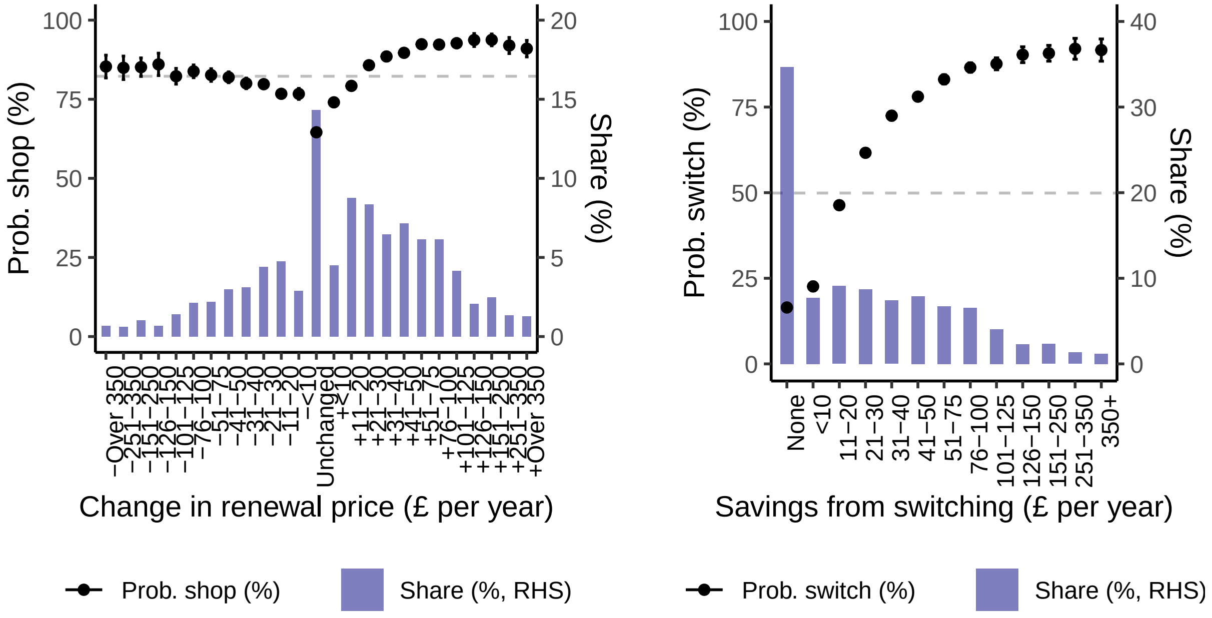
Displaying **25** of **25** available quotes

Your quotes explained

Provider	☑ Annual ☐ Monthly	Excesses	Personal accident	Courtesy car	Breakdown cover	Motor legal protection	Features Explained
Admiral Essential	£937.01	Compulsory £100 Voluntary £250 **Total** **£350**	✓ Included	✓ Included	**Add** Annually **£44.95**	**Add** Annually **£24.95**	**More Details**
This policy does not include windscreen and glass cover as standard							
GOGIRL	£957.60	Compulsory £200 Voluntary £250 **Total** **£450**	✓ Included	✓ Included	**Add** Annually **£42.99**	**Add** Annually **£14.99**	**More Details**
insure2drive	£959.84	Compulsory £200 Voluntary £250 **Total** **£450**	✓ Included	✓ Included	**Add** Annually **£42.99**	**Add** Annually **£14.99**	**More Details**

Notes: This figure shows the top of a menu of quotes for an insurance enquiry made via Compare the Market.

Figure B.1: Search and Switching



Notes: The left panel plots shopping probability against renewal price changes. Bars show the customer distribution across buckets; points show the fraction shopping in each bucket with 95% confidence intervals; the dashed line shows the average shopping probability. The right panel plots switching probability against potential savings for customers who shopped. Bars show the customer distribution; points show switching rates with 95% confidence intervals; the dashed line shows the average switching probability.

B Consumer Intelligence Survey Data

In this Appendix, we use *Consumer Intelligence* survey data to examine search and switching decisions at renewal. The survey covers individuals who renewed car insurance during July 2019 to December 2023. Respondents report their renewal price changes, whether they shopped around, potential savings from switching, and whether they switched insurers.

Figure B.1 summarizes search and switching patterns. The left panel shows how renewal price changes affect search decisions. Around 80% of customers report shopping around to some extent, though the survey does not reveal how comprehensively they search. Search responds to renewal price changes: customers facing large price increases are over 20 percentage points more likely to search. Surprisingly, customers receiving large price decreases also search slightly more.

The right panel shows how switching decisions depend on potential savings. Conditional on searching, around 50% of customers switch insurers. Switching rates increase steeply with savings: customers facing savings over £45 per year are over three times as likely to switch as those facing savings under £5.

The *Consumer Intelligence* sample differs from our main *Compare the Market* data. Table B.1 shows that *Compare the Market* includes younger drivers relative to both *Consumer Intelligence* and the UK driver population. We therefore cannot directly translate these survey responses into precise switching frequencies for our main sample. Nonetheless, the data support two conclusions: search and switching involve substantial

Table B.1: Datasets: shares by age

Age group	Compare the Market	Consumer Intelligence	Population
16-29	50	11	14
30-39	16	16	17
40-49	11	17	18
50-59	9	21	19
60-69	11	22	15
70+	3	13	17

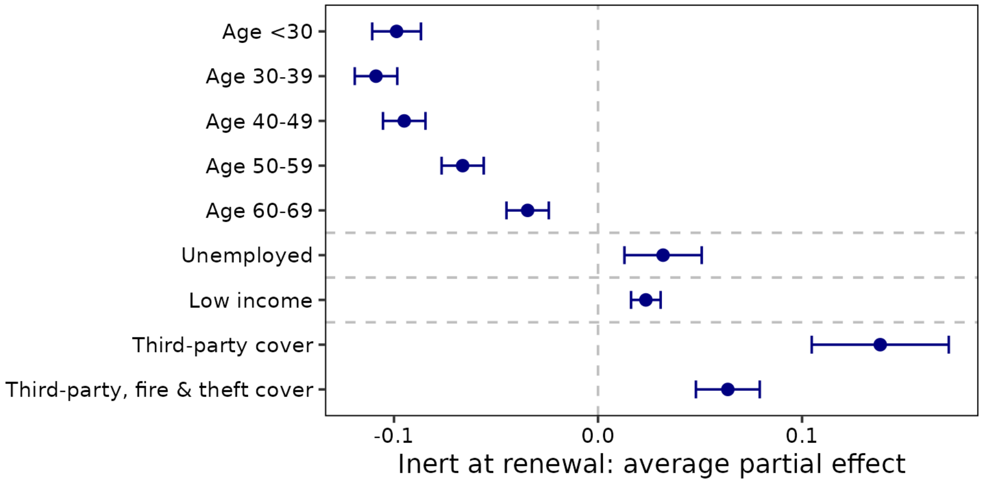
Notes: This table shows the sample shares by age group for our main dataset (Compare the Market), our renewals survey data (Consumer Intelligence) and the population of England & Wales based on national survey data.

frictions, yet customers' search and switching decisions respond to price incentives.

Figure B.2 summarizes search decisions as a function of customer demographics. We run probit regressions for whether a customer reported that they *did not* shop around at their last renewal as a function of their demographics, and plot the estimated average partial effects. The likelihood that a customer did not shop at their most recent renewal is higher for older customers, unemployed customers, low-income customers, and those shopping for non-comprehensive insurance.

Figure B.2 studies the *unconditional* probability of inertia across customer groups. Table 4, by contrast, studies the probability of inertia *conditional* on having searched at least once. Together, these results are consistent with older customers being less likely to search on average, but those older drivers who do search being more likely to do so every year.

Figure B.2: Survey Evidence on Customer Inertia



Notes: This figure summarizes survey evidence on customer inertia. The reported estimates are average marginal effects from probit regressions of a dummy for whether a customer reported not shopping around at their last renewal, based on *Consumer Intelligence* survey data. A customer is low-income if their income is in the bottom 20% of reported incomes. The omitted age category is over-70. The omitted coverage category is comprehensive insurance.

C Robustness

In Section 5.1 we construct proxies for customer inertia based on their demographics. To do so, we take all customers who search for and take out insurance in our data, and regress the likelihood that they *do not* search again 12 months later on their demographics. For example, a customer who takes out inertia in December 2019 is inert if they do not search again in December 2020. We find that this conditional likelihood of inertia is greater for young customers, disabled customers, those who are insuring cheap cars, and those who are searching for non-comprehensive insurance.

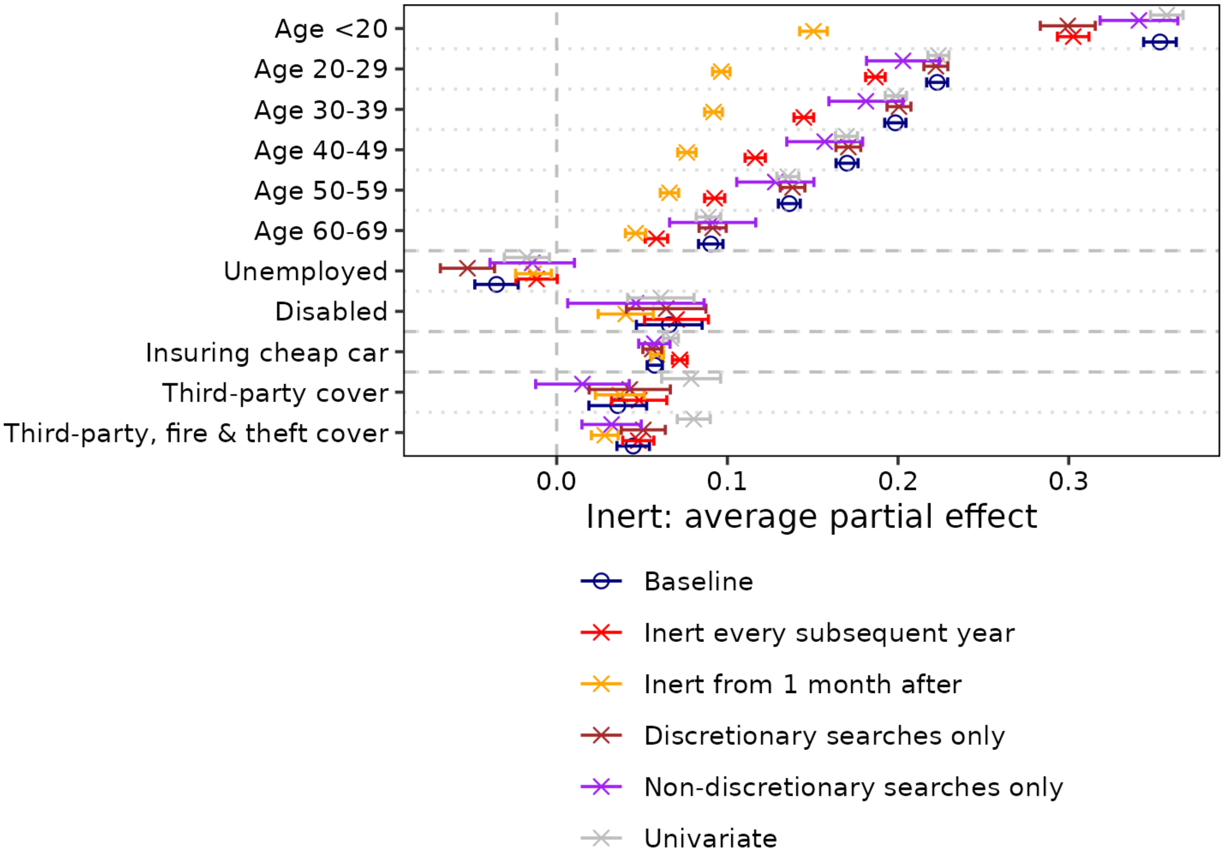
Figure C.1 shows the robustness of these patterns. Marginal effects from our baseline specification, as in Table 4, are shown in blue. We re-compute these marginal effects under the following assumptions:

1. We expand our definition of inertia to include every year after a customer initially shops. Thus, if a customer searches in December 2019, they are inert if and only if they are not seen to search in any subsequent December in our sample. We show these results in red.
2. We define a customer to be inert if they do not search again from one month after they initially take out insurance. So a customer who takes out insurance in December 2019 would be considered inert if and only if they do not search in any period from January 2020 onwards. We show these results in yellow.
3. We restrict the sample to initial searches that are in some sense discretionary, as they are not triggered by a customer buying a new car or newly acquiring a license. We show these results in brown.
4. We restrict the sample to initial searches that are *non-discretionary*, as they are follow a customer buying a new car or newly acquiring a license. That is, we include only those customers who were excluded in the previous test. We show these results in purple.
5. We include the demographics one-by-one in univariate regressions, rather than all together in a multivariate regression. We show these results in grey.

Across each of these specifications, the key patterns shown in Table 4 continue to hold.

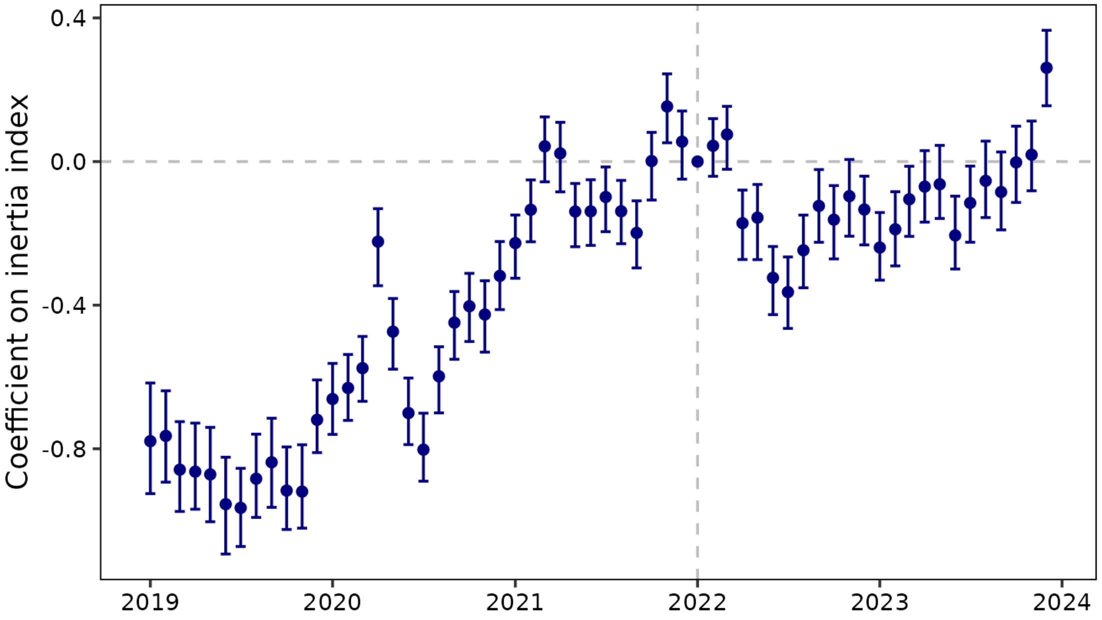
For completeness, we also replicate Figure 2 for the robustness test where we include only non-discretionary searches (purple points in Figure C.1). We show the results of this in Figure C.2 below. The results are very similar to the baseline, highlighting that the robustness of the relationship between inertia and demographics carries over into the relationship between prices and inertia.

Figure C.1: Inertia and Customer Characteristics: Robustness



Notes: This figure shows the robustness of our results on customer inertia as a function of customer characteristics. The reported estimates are average marginal effects from probit regressions of a dummy for whether a customer is inert after taking out insurance, for customers taking out insurance at least one year before the end of our sample. A car is cheap if it is in the bottom 20% of car values. 'Disabled' denotes a driver who is out of work due to illness or disability. The omitted age category is over-70. The omitted coverage category is comprehensive insurance. Our baseline results, as shown in Table 4, are in blue. The remaining colors show results under various robustness tests, as described in Section C.

Figure C.2: Pricing and Customer Inertia Through Time: Robustness



Notes: This figure shows the robustness of our results on the price of insurance and customer inertia, shown in Figure 2. We estimate customer inertia based only on initial searches that were non-discretionary, as described in Section C and shown in purple in Figure C.1. We then replicate the regression shown in Figure 2, where points show the coefficient on inertia each month. The reference month is January 2022. Error bars show 95% confidence intervals based on bootstrapping, where standard errors are clustered at the level of the enquirer.

D Proofs

Proof of Proposition 1.

We study existence and uniqueness separately.

Part 1: Existence

Using the expression for firm 0's market share among active consumers from equation (3), its lagged market share relative to an equal market split equals

$$y_{t+1} = \left[\frac{\Delta_t}{2} + \frac{p_t^1 - p_t^0}{2c} \right]_{-\frac{1}{2}}^{\frac{1}{2}}.$$

We ignore the truncation for now and verify it later.

Suppose firm 1 is playing the strategy in the statement of the proposition, so market shares evolve according to

$$y_{t+1} = \frac{\Delta_t}{2} + \frac{p_t^1 - p_t^0}{2c} = \frac{m + (1-a)\,\Delta_t - by_t}{2} - \frac{p_t^0}{2c}. \tag{D.1}$$

Note that firm 0's price coincides with the one in equation (12) if and only if

$$y_{t+1}(\Delta_t, y_t) = \left(\frac{1}{2} - a\right)\Delta_t - by_t. \tag{D.2}$$

Ignoring the truncation and price cap, firm 0's Bellman equation is

$$V_0(y_t, \Delta_t) = \begin{array}{l} \max_{p_t^0, y_{t+1}} \left\{ \left[(2-\lambda)\left(y_{t+1} + \frac{1}{2}\right) + \lambda\left(y_t + \frac{1}{2}\right) \right] p_t^0 + \delta E_t\left[V_0(y_{t+1}, \Delta_{t+1})\right] \right\} \\ \text{subject to } y_{t+1} = \frac{m + (1-a)\Delta_t - by_t}{2} - \frac{p_t^0}{2c} \end{array}. \tag{D.3}$$

Substituting the constraint into the objective to eliminate p_t^0 and rearranging gives

$$\begin{aligned} V_0(y_t, \Delta_t) = \max_{y_{t+1}} \Big\{ & c\big[1 + (2-\lambda)y_{t+1} + \lambda y_t\big] \\ & \times \big[m + (1-a)\Delta_t - by_t - 2y_{t+1}\big] + \delta E_t\big[V_0(y_{t+1}, \Delta_{t+1})\big] \Big\}. \end{aligned} \tag{D.4}$$

Analogously, ignoring the truncation and price cap, firm 1's Bellman equation is

$$\begin{aligned} V_1(y_t, \Delta_t) = \max_{y_{t+1}} \Big\{ & c\Big[(2-\lambda)\Big(\tfrac{1}{2} - y_{t+1}\Big) + \lambda\Big(\tfrac{1}{2} - y_t\Big)\Big] \\ & \times \Big[m + by_t - (1-a)\Delta_t + 2y_{t+1}\Big] + \delta E_t\big[V_1(y_{t+1}, \Delta_{t+1})\big] \Big\}. \end{aligned} \tag{D.5}$$

Moreover, firm 1's price coincides with the one in equation (12) if and only if (D.2) holds.

Lemma 1. *The Bellman functional equations* (D.4) *and* (D.5) *are solved by*

$$\begin{aligned} V_0(y, \Delta) &= A + By + C\Delta + Dy^2 + Ey\Delta + F\Delta^2, \\ V_1(y, \Delta) &= A - By - C\Delta + Dy^2 + Ey\Delta + F\Delta^2, \end{aligned} \tag{D.6}$$

where

$$A = \frac{cm + \delta F\sigma^2}{1-\delta}, \qquad B = \frac{cb(2-\lambda)(2-\delta b^2)}{2-\lambda(1-\delta)},$$
$$C = \frac{c}{2}\,\frac{4-\delta b^2}{3-\delta b^2}, \qquad D = c\,\frac{2-\lambda}{2}\,b^2,$$
$$E = c(2-\lambda)b\,\frac{(4-\delta b^2)(1-\delta b^2)}{4(3-\delta b^2)}, \qquad F = c(2-\lambda)\,\frac{4-\delta b^2}{8(3-\delta b^2)^2}.$$

Moreover, (D.2) *is the policy function associated with both Bellman equations.*

Proof. Since location shocks are symmetrically distributed across firms and independently distributed over time, we have $E[\Delta_{t+1}|\Delta_t, y_t] = 0$. Let $\sigma^2 := E[\Delta_{t+1}^2]$. Take expectation of (D.4) using dominated convergence to obtain

$$E_t\left[V_0(y, \Delta_{t+1})\right] = A + By + Dy^2 + F\sigma^2. \tag{D.7}$$

Substitute this expression in the objective function of the Bellman equation (D.4):

$$\begin{aligned} &c\big[1 + (2-\lambda)y_{t+1} + \lambda y_t\big]\big[m + (1-a)\Delta_t - by_t - 2y_{t+1}\big] \\ &\qquad + \delta\big(A + By_{t+1} + Dy_{t+1}^2 + F\sigma^2\big). \end{aligned} \tag{D.8}$$

We need to show that the policy function (D.2) maximizes this expression and that plugging this policy function into the objective (D.8) gives V_0 in (D.6).

Since the expression in (D.8) is strictly concave in y_{t+1}, the necessary and sufficient FOC for an interior optimum is:

$$c\left\{(2-\lambda)\left[m + (1-a)\,\Delta_t - by_t - 2y_{t+1}\right] - 2\left[1 + (2-\lambda)y_{t+1} + \lambda y_t\right]\right\} + \delta\left(B + 2Dy_{t+1}\right) = 0.$$

Substituting (D.2) and rearranging establishes optimality:

$$\begin{aligned} c\Big\{&(2-\lambda)\big[m + (1-a)\Delta_t - by_t\big] - 2\big(1 + \lambda y_t\big) \\ &- 4(2-\lambda)\big[\big(\tfrac{1}{2} - a\big)\Delta_t - by_t\big]\Big\} \\ &+ \delta\Big\{B + 2D\big[\big(\tfrac{1}{2} - a\big)\Delta_t - by_t\big]\Big\} = 0. \end{aligned}$$

where the equality follows from algebraic manipulations after substituting the expressions for a, b, and m.

Substitute (D.2) into the objective (D.8):

$$\begin{aligned} c\Big\{&1 + (2-\lambda)\big[\big(\tfrac{1}{2} - a\big)\Delta_t - by_t\big] + \lambda y_t\Big\} \\ &\times \Big\{m + (1-a)\Delta_t - by_t - 2\big[\big(\tfrac{1}{2} - a\big)\Delta_t - by_t\big]\Big\} \\ &+ \delta\big(A + By_{t+1} + Dy_{t+1}^2 + F\sigma^2\big). \end{aligned}$$

Algebraic manipulations verify that this expression coincides with V_0 in (D.6), so the proposed value function satisfies the Bellman functional equation.

The proof for firm 1 is analogous. Suppose firm 0 is playing the strategy in the statement of the proposition, so market shares evolve according to:

$$y_{t+1} = \frac{p_t^1}{2c} - \frac{m + by_t - (1-a)\,\Delta_t}{2}. \tag{D.9}$$

Firm 1's price coincides with the one in equation (12) if and only if equation (D.2) holds. Following the same steps as above, one shows that firm 1's Bellman equation (D.5) is solved by

$$V_1(y,\Delta) = A - By - C\Delta + Dy^2 + Ey\Delta + F\Delta^2.$$

Moreover, the associated policy function is (D.2), implying that it is optimal for firm 1 to set prices as in equation (12). $\square$

Verifying price cap and truncation

The prices that solve the unconstrained problem do not violate the price cap, since

$$p_t^0 = c\big(m + a\Delta_t + by_t\big) \le c\big(m + a\bar{\varepsilon} + \frac{b}{2}\big) \le K,$$

and

$$p_t^1 = c\big(m - a\Delta_t - by_t\big) \le c\big(m + a\bar{\varepsilon} + \frac{b}{2}\big) \le K,$$

where we used $|\Delta_t| \le \bar{\varepsilon}$, $|y_t| \le \frac{1}{2}$, and equation (11).

We now verify that the omitted truncation holds on the equilibrium path. With the strategies in (12), market shares evolve according to

$$y_{t+1} = \left[\frac{\Delta_t}{2} + \frac{p_t^1 - p_t^0}{2c}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \left[\left(\frac{1}{2} - a\right)\Delta_t - by_t\right]_{-\frac{1}{2}}^{\frac{1}{2}}. \tag{D.10}$$

Since the expression inside brackets is decreasing in y_t and increasing in Δ_t, to verify that the truncations do not bind, it suffices to evaluate the expression at the endpoints: $(y_t,\Delta_t) = \big(\frac{1}{2}, -\bar{\varepsilon}\big)$ and $(y_t,\Delta_t) = \big(-\frac{1}{2}, \bar{\varepsilon}\big)$. In both cases, we have

$$|y_{t+1}| \le \frac{1}{2} \iff \left(\frac{1}{2} - a\right)\bar{\varepsilon} + \frac{b}{2} \le \frac{1}{2},$$

which holds due to (10) and the definition of a. Therefore, market share truncations do not bind along the equilibrium path.

Next, we show that there are no profitable deviations to off-path market shares where the truncation binds. If firm 0 sets prices so high that it does not sell to any new customers ($y_{t+1} = -\frac{1}{2}$) or if it sets prices so low that it sells to all new customers ($y_{t+1} = \frac{1}{2}$), the truncations bind and equation (D.10) no longer simplifies to (D.1). In particular, equation (D.10) does not uniquely pin down firm 0's price, so we cannot substitute the price out in Bellman equation (D.4). Therefore, we need to explicitly consider the payoff from deviating to prices in those boundary market shares.

Define two threshold prices:

$$p_t^+ := c\big[m - 1 + (1-a)\Delta_t - by_t\big], \qquad p_t^- := c\big[m + 1 + (1-a)\Delta_t - by_t\big].$$

Under firm 1's fixed strategy, these are the prices that induce $y_{t+1} = 1/2$ and $y_{t+1} = -1/2$, respectively. That is, firm 0 gets all new consumers if $p_t^0 \le p_t^+$, the market share is interior if $p_t^0 \in (p_t^+, p_t^-)$, and firm 0 gets no new consumers if $p_t^0 \ge p_t^-$. To establish that the market share truncations do not bind off the equilibrium path, we need to show that setting prices below p_t^+ or above p_t^- is not optimal.

We first show that firm 0 never profits by choosing a price below p_t^+. Intuitively, if the firm sets prices so low that it sells to all new customers ($y_{t+1} = \frac{1}{2}$), further reductions in price do not translate into higher sales, thereby reducing profits. Note that $\Delta_t \leq \bar{\varepsilon}$ and $y_t \leq \frac{1}{2}$ imply

$$p_t^+ \leq c\left[m - 1 + (1-a)\bar{\varepsilon} + \frac{b}{2}\right].$$

We claim that

$$m - 1 + (1-a)\bar{\varepsilon} + \frac{b}{2} < m + a\bar{\varepsilon} + \frac{b}{2}.$$

To see this, subtract the LHS from the RHS

$$\left(m + a\bar{\varepsilon} + \frac{b}{2}\right) - \left[m - 1 + (1-a)\bar{\varepsilon} + \frac{b}{2}\right] = 1 - (1-2a)\bar{\varepsilon} > 0,$$

where the inequality substitutes the definition of a in equation (10) and uses $b \in (0,1)$. Hence

$$p_t^+ < c\left(m + a\bar{\varepsilon} + \frac{b}{2}\right) \leq K.$$

Therefore, firm 0's best deviation when capturing the entire market ($y_{t+1} = \frac{1}{2}$) is obtained by setting price p_t^+. Writing G_t for the objective function of firm 0's Bellman equation (D.4), the payoff from choosing price p_t^+ equals $G_t(1/2)$. Since G_t is strictly concave in $[-1/2, 1/2]$ with an interior optimum (D.2), all deviations to prices below p_t^+ are dominated by the interior optimum.

Next, consider deviations that lead to $y_{t+1} = -\frac{1}{2}$. Such deviations are achieved by setting a price above p_t^-, which are high enough that no new consumer buys. There are two cases. If $K < p_t^-$, then no feasible price induces $y_{t+1} = -\frac{1}{2}$, since setting a price above K violates the price cap. Hence there is no feasible deviation to $y_{t+1} = -\frac{1}{2}$ in this case.

If instead $K \geq p_t^-$, then every price $p_t^0 \in [p_t^-, K]$ induces $y_{t+1} = -\frac{1}{2}$. Conditional on choosing a price in this interval, the continuation value is $W_0(-1/2)$ (constant) and the current payoff is $\lambda\big(y_t + \frac{1}{2}\big)p_t^0$, which is increasing in p_t^0. Hence the best deviation conditional on inducing y_{t+1} is $p_t^0 = K$. That is, since the firm does not sell to any new customer, it only faces passive consumers for whom it is optimal to charge the highest price allowed. Define the profit from this deviation as:

$$\Pi_t^K := \lambda\Big(y_t + \frac{1}{2}\Big)K + \delta W_0\Big(-\frac{1}{2}\Big). \tag{D.11}$$

Since p_t^- is the threshold price that also yields $y_{t+1} = -1/2$,

$$G_t\Big(-\frac{1}{2}\Big) = \lambda\Big(y_t + \frac{1}{2}\Big)p_t^- + \delta W_0\Big(-\frac{1}{2}\Big).$$

Substitute in (D.11) to rewrite the payoff from the deviation as

$$\Pi_t^K = G_t\Big(-\frac{1}{2}\Big) + \lambda\Big(y_t + \frac{1}{2}\Big)\left(K - p_t^-\right).$$

Because G_t is a quadratic function with (constant) second derivative $-c(2-\lambda)(4-\delta b^2)$ and is maximized at $y_{t+1}(\Delta_t, y_t)$,

$$G_t(z) = G_t\left(y_{t+1}(\Delta_t, y_t)\right) - \frac{c(2-\lambda)(4-\delta b^2)}{2}\left[z - y_{t+1}(\Delta_t, y_t)\right]^2 \qquad \forall z \in \mathbb{R}.$$

Evaluating at $z = -1/2$ gives

$$G_t\left(y_{t+1}(\Delta_t, y_t)\right) - G_t\Big(-\frac{1}{2}\Big) = \frac{c(2-\lambda)(4-\delta b^2)}{2}\left[y_{t+1}(\Delta_t, y_t) + \frac{1}{2}\right]^2.$$

Since $V_0(y_t, \Delta_t) = G_t(y_{t+1}(\Delta_t, y_t))$ by the Bellman equation (D.4), we obtain

$$V_0(y_t, \Delta_t) - \Pi_t^K = \frac{c(2-\lambda)(4-\delta b^2)}{2}\left[y_{t+1}(\Delta_t, y_t) + \frac{1}{2}\right]^2 - \lambda\Big(y_t + \frac{1}{2}\Big)\big(K - p_t^-\big).$$

Substituting the expression from (D.2) and using $\Delta_t \geq -\bar{\varepsilon}$ and $y_t \leq \frac{1}{2}$, gives

$$y_{t+1}(\Delta_t, y_t) + \frac{1}{2} = \frac{1}{2} + \left(\frac{1}{2} - a\right)\Delta_t - by_t \geq \frac{1}{2} - \left(\frac{1}{2} - a\right)\bar{\varepsilon} - \frac{b}{2} \geq 0,$$

and

$$p_t^- = c\big[m + 1 + (1-a)\Delta_t - by_t\big] \geq c\left[m + 1 - (1-a)\bar{\varepsilon} - \frac{b}{2}\right].$$

Then, since $y_t + \frac{1}{2} \leq 1$, we have

$$\begin{aligned} V_0(y_t, \Delta_t) - \Pi_t^K \geq{}& \frac{c(2-\lambda)(4-\delta b^2)}{2}\left[\frac{1}{2} - \Big(\frac{1}{2} - a\Big)\bar{\varepsilon} - \frac{b}{2}\right]^2 \\ &- \lambda\left[K - c\Big(m + 1 - (1-a)\bar{\varepsilon} - \frac{b}{2}\Big)\right]. \end{aligned}$$

By the upper bound on K from (11), the RHS is non-negative. Hence, $p_t^0 = K$ is never a profitable deviation.

To summarize, we have shown that it is not optimal for firm 0 to choose any price outside $[p_t^+, p_t^-]$ (where the truncation in market shares binds). The same argument applies to firm 1 by symmetry.

Part 2: Uniqueness

Next, we verify that the MPE above is the unique MPE in affine strategies with interior market shares. Suppose firms 0 and 1 choose affine prices:

$$\begin{aligned} p^0(\Delta_t, y_t) &= c\big[m - \mu + (a+h)\Delta_t + (b+k)y_t\big], \\ p^1(\Delta_t, y_t) &= c\big[m + \mu + (h-a)\Delta_t + (k-b)y_t\big], \end{aligned} \tag{D.12}$$

for some a, b, m (not necessarily equal to the ones in equations (8)-(9)) and some μ, h, k. These expressions decompose the price into symmetric (a, b, m) and asymmetric (μ, h, k) components. Note that any affine price strategy can be written as above. In particular, firms follow symmetric pricing strategies if $\mu = h = k = 0$.

As in the existence part, we ignore the truncation for now and verify it later. Fix firm 1's affine strategy from (D.12). Substituting firm 1's strategy in the law of motion for market shares, gives

$$y_{t+1} = \frac{\Delta_t}{2} + \frac{p_t^1 - p_t^0}{2c} = \frac{m + \mu + (1+h-a)\Delta_t + (k-b)y_t}{2} - \frac{p_t^0}{2c} \tag{D.13}$$

Solving for prices and substituting the law of motion into the objective, we obtain firm 0's Bellman equation:

$$V_0(y_t, \Delta_t) = \max_{y_{t+1}}\Big\{c\big[1 + (2-\lambda)y_{t+1} + \lambda y_t\big]\big[m + \mu + (1+h-a)\Delta_t + (k-b)y_t - 2y_{t+1}\big] + \delta W_0(y_{t+1})\Big\},$$

where we write $W_0(y_{t+1}) := E\big[V_0(y_{t+1}, \Delta_{t+1})\big]$ to simplify notation.

The first-order condition is

$$c\Big[(2-\lambda)\big(m+\mu+(1+h-a)\Delta_t+(k-b)y_t-2y_{t+1}\big)-2\big(1+(2-\lambda)y_{t+1}+\lambda y_t\big)\Big]+\delta W_0'(y_{t+1})=0.$$

Differentiating the Bellman equation with respect to y_t gives the envelope condition

$$\frac{\partial V_0}{\partial y}(y_t,\Delta_t)=c\left\{\lambda\left[m+\mu+(1+h-a)\Delta_t+(k-b)y_t-2y_{t+1}\right]+(k-b)\big[1+(2-\lambda)y_{t+1}+\lambda y_t\big]\right\}.$$

By assumption, it is optimal for firm 0 to choose the affine strategy in (D.12). Substituting in the expression above, gives

$$\frac{\partial V_0}{\partial y}(y_t,\Delta_t)=cA_0+c\left[\lambda(a+h)+(k-b)(2-\lambda)\left(\frac{1}{2}-a\right)\right]\Delta_t+cB_0y_t, \tag{D.14}$$

where

$$A_0:=\lambda(m-\mu)+(k-b)\left[1+(2-\lambda)\mu\right],$$

and

$$B_0:=\lambda(b+k)+(k-b)\left[\lambda-(2-\lambda)b\right]=(2-\lambda)b^2+k\left[2\lambda-(2-\lambda)b\right].$$

Equation (D.14) holds for all $(\Delta_t, y_t)\in(-\bar{\varepsilon},\bar{\varepsilon})\times(-1/2,1/2)$. Since the right-hand side is bounded on the compact state space, dominated convergence yields

$$W_0'(y)=E\left[\frac{\partial V_0}{\partial y}(y,\Delta_{t+1})\right]=cA_0+cB_0y, \qquad y\in\left(-\frac{1}{2},\frac{1}{2}\right),$$

where we used $E[\Delta_{t+1}]=0$.

Next, consider the optimality conditions for firm 1. As before, substitute firm 0's affine strategy in the law of motion for market shares:

$$p_t^1=p^0(\Delta_t,y_t)-c\Delta_t+2cy_{t+1}=c\Big[m-\mu+(a+h-1)\Delta_t+(b+k)y_t+2y_{t+1}\Big].$$

Hence firm 1's Bellman equation is

$$V_1(y_t,\Delta_t)=\max_{y_{t+1}}\Big\{c\big[1-(2-\lambda)y_{t+1}-\lambda y_t\big]\big[m-\mu+(a+h-1)\Delta_t+(b+k)y_t+2y_{t+1}\big]+\delta W_1(y_{t+1})\Big\},$$

with first-order condition

$$\begin{aligned}c\Big\{&-(2-\lambda)\big[m-\mu+(a+h-1)\Delta_t+(b+k)y_t+2y_{t+1}\big]\\&+2\big[1-(2-\lambda)y_{t+1}-\lambda y_t\big]\Big\}+\delta W_1'(y_{t+1})=0,\end{aligned}$$

where $W_1(y):=E\big[V_1(y,\Delta_{t+1})\big]$. The envelope condition is

$$\frac{\partial V_1}{\partial y}(y_t,\Delta_t)=-\lambda p_t^1+c(b+k)\left[1-(2-\lambda)y_{t+1}-\lambda y_t\right].$$

By assumption, it is optimal for firm 1 to choose the affine strategy in (D.12). Substituting in the expression above, gives

$$\frac{\partial V_1}{\partial y}(y_t,\Delta_t)=cA_1+c\left[-\lambda(h-a)-(b+k)(2-\lambda)\left(\frac{1}{2}-a\right)\right]\Delta_t+cB_1y_t,$$

where

$$A_1 := -\lambda(m+\mu) + (b+k)\big(1-(2-\lambda)\mu\big),$$

and

$$B_1 := -\lambda(k-b) + (b+k)\left[(2-\lambda)b-\lambda\right] = (2-\lambda)b^2 + k\left[(2-\lambda)b - 2\lambda\right].$$

Again, by dominated convergence and $E[\Delta_{t+1}] = 0$,

$$W_1'(y) = E\left[\frac{\partial V_1}{\partial y}(y, \Delta_{t+1})\right] = cA_1 + cB_1 y, \qquad y \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Under the affine strategies for both firms in (D.12), market shares evolve according to

$$y_{t+1} = \frac{\Delta_t}{2} + \frac{p_t^1 - p_t^0}{2c} = \mu + \left(\frac{1}{2} - a\right)\Delta_t - by_t. \tag{D.15}$$

Substituting in the two first-order conditions and matching the coefficients on the constant, Δ_t, and y_t, gives six equations:

$$(2-\lambda)(m+\mu) - 2 - 4(2-\lambda)\mu + \delta(A_0 + B_0\mu) = 0, \tag{A1}$$

$$(2-\lambda)(1+h-a) - 4(2-\lambda)\left(\frac{1}{2} - a\right) + \delta B_0\left(\frac{1}{2} - a\right) = 0, \tag{A2}$$

$$(2-\lambda)(k+3b) - 2\lambda - \delta B_0 b = 0, \tag{A3}$$

$$2 - (2-\lambda)(m-\mu) - 4(2-\lambda)\mu + \delta(A_1 + B_1\mu) = 0, \tag{A4}$$

$$(2-\lambda)(1-a-h) - 4(2-\lambda)\left(\frac{1}{2} - a\right) + \delta B_1\left(\frac{1}{2} - a\right) = 0, \tag{A5}$$

$$(2-\lambda)(3b-k) - 2\lambda - \delta B_1 b = 0. \tag{A6}$$

Rearranging the equations above yields the equivalent system:[D.1]

$$(2-\lambda)\delta b^3 - 3(2-\lambda)b + 2\lambda = 0, \tag{B1}$$

$$2a(3-\delta b^2) = 2 - \delta b^2, \tag{B2}$$

$$k\left[(2-\lambda)(1+\delta b^2) - 2\delta\lambda b\right] = 0, \tag{B3}$$

$$(2-\lambda)h + \delta\left(\frac{1}{2} - a\right)k\left[2\lambda - (2-\lambda)b\right] = 0, \tag{B4}$$

$$m\left[2 - \lambda(1-\delta)\right] - 2 - \delta b + \delta k\mu\left[2 + \lambda - b(2-\lambda)\right] = 0, \tag{B5}$$

$$\delta k - \mu\left\{6 + 2\delta b(1-b) + \lambda\left[\delta(b^2 - b + 1) - 3\right]\right\} = 0. \tag{B6}$$

We claim that this system has a unique solution, which coincides with the symmetric MPE obtained previously.

Since $y_{t+1} \in [-\frac{1}{2}, \frac{1}{2}]$, evaluating the law of motion (D.15) at $\Delta_t = 0$ and $y_t = \pm\frac{1}{2}$, gives

$$|\mu| + \frac{|b|}{2} \leq \frac{1}{2} \therefore |b| \leq 1.$$

[D.1] Specifically, (B1) is obtained by combining (A3) and (A6); (B2) combines (A2) and (A5); (B3) combines (A6) and (A3); (B4) combines (A5) and (A2); (B5) combines (A4) and (A1); and finally (B6) combines (A1) and (A4).

The first equation can be written as

$$f(b) := \delta b^3 - 3b + \frac{2\lambda}{2-\lambda} = 0.$$

Differentiation establishes that $f'(b) = 3(\delta b^2 - 1) < 0$ for $b \in [-1, 1]$, and

$$f(-1) = 3 - \delta + \frac{2\lambda}{2-\lambda} > 0, \qquad f(0) = \frac{2\lambda}{2-\lambda} > 0, \qquad f(1) = \delta - 3 + \frac{2\lambda}{2-\lambda} < 0.$$

Therefore f has a unique root in $(-1, 1)$, and since $f(0) > 0 > f(1)$, that root lies in $(0, 1)$. Thus $b \in (0, 1)$ is uniquely determined.

The second equation gives

$$a = \frac{2 - \delta b^2}{2(3 - \delta b^2)}.$$

Next, consider the third equation. We claim that the term inside the square bracket is strictly positive. To see this, note that since

$$\frac{2(1 + \delta b^2)}{(1 + \delta b^2) + 2\delta b} > 1 \iff b^2 - 2b + \frac{1}{\delta} > 0,$$

which is true since $b^2 - 2b + \frac{1}{\delta}$ is an upward facing parabola with a minimum of $\frac{1}{\delta} - 1 > 0$ at $b = 1$ (where we used $\delta < 1$). Hence, the third equation gives $k = 0$.

Substitute $k = 0$ in the fourth and fifth equations:

$$(2 - \lambda)h = 0 \therefore h = 0,$$

$$m\left[2 - \lambda(1 - \delta)\right] = 2 + \delta b \therefore m = \frac{2 + \delta b}{2 - \lambda(1 - \delta)}.$$

Finally, substituting $k = 0$ in the sixth equation, gives

$$\mu \left\{6 + 2\delta b(1 - b) + \lambda \left[\delta(b^2 - b + 1) - 3\right]\right\} = 0.$$

We claim that the term inside brackets is strictly positive. To see this, note that $b \in (0, 1)$ implies

$$2\delta b(1 - b) \geq 0, \qquad \delta(b^2 - b + 1) > 0,$$

and therefore

$$6 + 2\delta b(1 - b) + \lambda \left[\delta(b^2 - b + 1) - 3\right] > 6 - 3\lambda > 0.$$

Hence $\mu = 0$.

We have shown that any affine interior MPE must satisfy $\mu = h = k = 0$ and therefore must be symmetric, concluding the proof.

Verifying price cap and truncation

Let $S := [-\bar{\varepsilon}, \bar{\varepsilon}] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ denote the state space and let $S^\circ := (-\bar{\varepsilon}, \bar{\varepsilon}) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ denote its interior. Suppose firms 0 and 1 use affine prices:

$$\begin{aligned} p^0(\Delta_t, y_t) &= c\big[m - \mu + (a + h)\Delta_t + (b + k)y_t\big], \\ p^1(\Delta_t, y_t) &= c\big[m + \mu + (h - a)\Delta_t + (k - b)y_t\big]. \end{aligned} \tag{D.16}$$

Define the "non-truncated" law of motion,

$$g(\Delta_t, y_t) := \frac{\Delta_t}{2} + \frac{p^1(\Delta_t, y_t) - p^0(\Delta_t, y_t)}{2c} = \mu + \left(\frac{1}{2} - a\right)\Delta_t - by_t.$$

The law of motion for y_{t+1} corresponds to its truncation $[g(\Delta_t, y_t)]_{-\frac{1}{2}}^{\frac{1}{2}}$.

Incorporating the price cap and truncation, firm 0's Bellman equation is

$$\begin{aligned} V_0(y_t, \Delta_t) &= \max_{y_{t+1}, p_t^0 \le K} \left\{ \left[(2-\lambda)\left(y_{t+1} + \tfrac{1}{2}\right) + \lambda\left(y_t + \tfrac{1}{2}\right)\right] p_t^0 + \delta W_0\left(y_{t+1}\right) \right\} \\ &\text{subject to } y_{t+1} = \left[\tfrac{\Delta_t}{2} + \tfrac{p^1(\Delta_t, y_t) - p_t^0}{2c}\right]_{-\frac{1}{2}}^{\frac{1}{2}}, \end{aligned}, \tag{D.17}$$

where $W_0(y) := E[V_0(y, \Delta_{t+1})]$. Firm 1's Bellman equation is defined analogously. As usual, we write $p^i \equiv K$ for $p^i(y_t, \Delta_t) = K$ for all $(y_t, \Delta_t) \in S$.

Lemma 2. *Fix an MPE with affine strategies and suppose $p^i \not\equiv K$ for $i \in \{0, 1\}$. Then $y_{t+1}(\Delta, y) \in (-\frac{1}{2}, \frac{1}{2})$ in any non-empty open set.*

Proof. If, on a non-empty open set of states, a firm chooses the lower boundary $y_{t+1} = -\frac{1}{2}$, then on that set its optimal price is K (as argued in the uniqueness part, all prices above the threshold induce the same allocation and continuation value, so the highest feasible price maximizes profits). Since the price function is affine, equality to a constant K on a nonempty open set implies equality to K in all states.

If, on a non-empty open set of states, a firm chooses the upper boundary $y_{t+1} = \frac{1}{2}$, then on that set it must choose the threshold price

$$p_t^{++} := p^1(\Delta_t, y_t) - (1 - \Delta_t)\, c = [m + \mu + (h - a)\Delta_t + (k - b)y_t - (1 - \Delta_t)]\, c,$$

(as in the uniqueness part, lowering prices further does not attract new customers but reduces profits). Since the price strategy and p_t^{++} are both affine, agreement on a nonempty open set implies agreement in all states. But then the rival does not sell to any new consumer in any state, so its optimal price is K in all states (by the argument from the previous paragraph since firms are symmetric).

Thus in any affine MPE y_{t+1} is at a boundary on a non-empty open set, one firm must set the constant price $p^i \equiv K$. □

Lemma 3. *Fix an MPE with affine strategies and suppose $p^i \not\equiv K$ for $i \in \{0, 1\}$. Then*

$$p^0(\Delta, y) < K, \qquad p^1(\Delta, y) < K, \qquad |g(\Delta, y)| < \frac{1}{2} \qquad \textit{for all } (\Delta, y) \in S^\circ.$$

Proof. For $i = 0, 1$, define the slack function $L_i : S \to \mathbb{R}_+$ associated with the price cap

$$L_i(\Delta, y) := K - p^i(\Delta, y).$$

Since p^i is affine and $p^i(\Delta, y) \le K$ in every state, L_i is a nonnegative, affine function on S. If $L_i(\Delta_0, y_0) = 0$ at some $(\Delta_0, y_0) \in S^\circ$, then L_i attains its minimum at an interior point of the open set S°. Since L_i is affine, this implies

$$0 = \nabla L_i(\Delta_0, y_0) = \nabla L_i(\Delta, y) \text{ for all } (\Delta, y) \in S$$

(affine functions with an interior minimum over a convex domain must be constant). Hence L_i is constant and equal to zero. Therefore, $p^i \equiv K$, a contradiction. Thus

$$p^0(\Delta, y) < K \quad \text{and} \quad p^1(\Delta, y) < K \quad \text{for all} \quad (\Delta, y) \in S^\circ.$$

Next suppose that $g(\Delta_0, y_0) > \frac{1}{2}$ at some $(\Delta_0, y_0) \in S^\circ$ (the upper truncation binds at some interior point). By continuity of the affine function g, there is a nonempty open set $O \subset S^\circ$ on which $g(\Delta, y) > \frac{1}{2}$. Therefore, the upper truncation binds at O, meaning that at every state in O, the following period's market share is $y_{t+1} = \frac{1}{2}$. If firm 1 raises its current price to K, the truncation still yields $y_{t+1} = \frac{1}{2}$, keeping both the allocation and the continuation value unchanged. Firm 1's current profit increases by

$$\lambda \left(\frac{1}{2} - y \right) \left(K - p^1(\Delta, y) \right) > 0,$$

because $y < \frac{1}{2}$ on S° and $p^1(\Delta, y) < K$ on S°. Hence optimality requires $p^1(\Delta, y) = K$ on O. Since p^1 is affine and K is a constant, equality on a nonempty open set implies equality for all points, $p^1 \equiv K$, a contradiction. Therefore

$$g(\Delta, y) \le \frac{1}{2} \qquad \forall (\Delta, y) \in S^\circ.$$

If $g(\Delta_0, y_0) = \frac{1}{2}$ at some $(\Delta_0, y_0) \in S^\circ$, then the affine function $\frac{1}{2} - g$, which is nonnegative on S° by the previous paragraph, vanishes at an interior point. Hence,

$$g(\Delta, y) = \frac{1}{2} \qquad \forall (\Delta, y) \in S^\circ.$$

Fix any $(\Delta, y) \in S^\circ$. Since $p^1(\Delta, y) < K$, there exists $\eta > 0$ such that $p^1(\Delta, y) + \eta \le K$. Deviating only at that state to price $p^1(\Delta, y) + \eta$ still yields $y_{t+1} = \frac{1}{2}$, so the continuation value remains unchanged, while current profit increases by $\lambda \left(\frac{1}{2} - y \right) \eta > 0$, a contradiction. Hence,

$$g(\Delta, y) < \frac{1}{2} \qquad \forall (\Delta, y) \in S^\circ.$$

The proof that $g(\Delta, y) > -\frac{1}{2}$ on S° is symmetric: if $g < -\frac{1}{2}$ on a nonempty open set, then firm 0 can raise its price to K without changing allocation or continuation value, which contradicts $p^0 \not\equiv K$. If $g(\Delta, y) = -\frac{1}{2}$ at an interior point, then affine-ness implies $g \equiv -\frac{1}{2}$ on S°, and a small upward deviation by firm 0 is profitable. Therefore,

$$|g(\Delta, y)| < \frac{1}{2} \qquad \forall (\Delta, y) \in S^\circ.$$

□

Lemma 4. *There is no MPE with affine strategies in which $p^i \equiv K$ for some $i \in \{0, 1\}$.*

Proof. By symmetry, it suffices to rule out an affine MPE with $p^1 \equiv K$. Let $p^0(\Delta, y)$ denote firm 0's affine price and define the non-truncated law of motion after substituting firm 1's strategy as:

$$h(\Delta, y) := \frac{\Delta}{2} + \frac{K - p^0(\Delta, y)}{2c} = \alpha + \beta\Delta - \rho y.$$

Because $p^0(\Delta, y) \le K$ for all (Δ, y) and $\bar{\varepsilon} \le \frac{1}{2}$,

$$h(\Delta, y) \ge \frac{\Delta}{2} \ge -\frac{\bar{\varepsilon}}{2} > -\frac{1}{2}.$$

Hence the lower truncation never binds. Incorporating the upper truncation gives the law of motion for market shares:

$$y' = \min \left\{ h(\Delta, y), \frac{1}{2} \right\}.$$

We first rule out $p^0 \equiv K$. Suppose $p^0 \equiv K$ and consider the state $(y, \Delta) = (-\frac{1}{2}, 0)$. Under $p^0 \equiv p^1 \equiv K$, next period's market shares equal $y' = 0$, so firm 0's current-period payoff is

$$\frac{2-\lambda}{2}K.$$

Suppose firm 0 deviates at that state to $p^0 = K - \eta$, where $0 < \eta < \min\{c, K-c\}$. Then, $y' = \frac{\eta}{2c} \in \left(0, \frac{1}{2}\right)$, so firm 0's current-period payoff becomes

$$(2-\lambda)\left(\tfrac{1}{2} + \frac{\eta}{2c}\right)(K-\eta).$$

Hence the current-profit gain is

$$\frac{2-\lambda}{2}\eta\left(\frac{K}{c} - 1 - \frac{\eta}{c}\right) > 0,$$

which is strictly positive because $K > c$. By the one-shot deviation principle, from date $t+1$ onward the firms revert to the original Markovian strategy $p^0 \equiv p^1 \equiv K$. Under that strategy, the market share at date $t+2$ reverts back to

$$y_{t+2} = \frac{\Delta_{t+1}}{2},$$

so the deviation affects continuation values only through date $t+1$. At date $t+1$, the deviation gives firm 0 an additional stock of $\eta/(2c)$ passive customers, which yields an additional renewal profit of

$$\delta\lambda K\frac{\eta}{2c} > 0.$$

Therefore the deviation gives firm 0 a profit in periods t and $t+1$, while keeping payoffs in all subsequent periods unchanged, contradicting optimality. Hence

$$p^0 \not\equiv K.$$

Next we show that $h(\Delta, y) \leq \frac{1}{2}$ for all states. Suppose instead that $h(\Delta_0, y_0) > \frac{1}{2}$ for some (Δ_0, y_0) which, using the definition of h, can be written as

$$p^0(\Delta_0, y_0) < K - c(1-\Delta_0).$$

Consider a deviation to $\tilde{p}^0(\Delta_0, y_0) = K - c(1-\Delta_0)$. Both $p^0(\Delta_0, y_0)$ and $\tilde{p}^0(\Delta_0, y_0)$ give the same future stock $y' = \frac{1}{2}$ and hence the same continuation value and current demand in both cases equals

$$(2-\lambda) + \lambda\left(y + \frac{1}{2}\right) > 0,$$

so the deviation increases current profits by:

$$\left[(2-\lambda) + \lambda\left(y + \frac{1}{2}\right)\right]\left[\tilde{p}^0(\Delta_0, y_0) - p^0(\Delta_0, y_0)\right] > 0,$$

contradicting optimality.

If $h \equiv \frac{1}{2}$, then by definition

$$p^0(\Delta, y) = K - c(1-\Delta) \qquad \text{for all } (\Delta, y).$$

Consider the state $(y, \Delta) = (\frac{1}{2}, 0)$. Under $(p^0, p^1) = (K-c, K)$, firm 1 has no locked-in consumers and gets no new consumers, so its current profit is 0. If instead firm 1 deviates only at that state to

$$p^1 = K - \eta, \qquad 0 < \eta < c,$$

then

$$y' = \frac{(K-\eta)-(K-c)}{2c} = \frac{1}{2} - \frac{\eta}{2c},$$

so current profit becomes

$$(2-\lambda)\frac{\eta}{2c}(K-\eta) > 0.$$

By the one-shot deviation principle, from date $t+1$ onward the firms revert to the original strategies $(K - c(1-\Delta), K)$. Under that profile, market shares at date $t+2$ are again $\frac{1}{2}$, independently of y_{t+1}, so the only future effect of the deviation is at date $t+1$. The deviation gives firm 1 an additional passive stock $\eta/(2c)$ at date $t+1$, generating extra renewal profit

$$\delta\lambda K\frac{\eta}{2c} > 0.$$

Therefore this deviation is strictly profitable, a contradiction.

Therefore $h \leq \frac{1}{2}$ and $h \not\equiv \frac{1}{2}$, which, because h is affine (and therefore continuous), implies

$$U := \left\{(\Delta, y) \in S^\circ : \ h(\Delta, y) < \frac{1}{2}\right\}$$

is a non-empty open set. Also, since $p^0 \not\equiv K$ and p^0 is affine with $p^0 \leq K$ everywhere, we must have

$$p^0(\Delta, y) < K \qquad \forall(\Delta, y) \in S^\circ.$$

Indeed, if $p^0(\Delta_0, y_0) = K$ at some interior point, then $K - p^0$ is a nonnegative affine function with an interior zero, hence it is identically zero, contradicting $p^0 \not\equiv K$ (affine functions over a convex domain with interior minima must be constant).

Thus, on U, the solution to firm 0's Bellman problem (D.17) is interior: the price cap is slack and the law of motion is non-truncated. Moreover, since the lower truncation of the law of motion never binds, the law of motion is globally affine

$$y' = h(\Delta, y) = \alpha + \beta\Delta - \rho y.$$

Now consider firm 0's evaluation operator under the fixed strategy profile (p^0, K):

$$(\mathcal{T}_0\Phi)(y, \Delta) = \Big[1 + (2-\lambda)h(\Delta, y) + \lambda y\Big]\Big[K + c\Delta - 2c\, h(\Delta, y)\Big] + \delta E\big[\Phi(h(\Delta, y), \Delta')\big].$$

Let $\mathcal{Q}$ denote the six-dimensional space of quadratic polynomials in (y, Δ). Because h is affine, $\mathcal{T}_0$ maps $\mathcal{Q}$ into itself. Since $\mathcal{T}_0$ is a contraction, its unique fixed point, namely firm 0's continuation value V_0, belongs to $\mathcal{Q}$. Since V_0 is quadratic, its derivative $\frac{\partial V_0}{\partial y}$ is affine, and

$$W_0'(y) := \frac{d}{dy}E[V_0(y, \Delta')]$$

is also affine in y.

On U, write firm 0's constrained Bellman problem with $z = y'$ as the choice variable:

$$V_0(y, \Delta) = \max_{z \in [\Delta/2,\, 1/2]} \left\{\Big[1 + (2-\lambda)z + \lambda y\Big]\Big[K + c\Delta - 2cz\Big] + \delta W_0(z)\right\}.$$

At states in U, the equilibrium choice $z = h(\Delta, y)$ lies strictly inside $[\Delta/2, 1/2]$ because $p^0 < K$ implies $h(\Delta, y) > \Delta/2$, while $h(\Delta, y) < 1/2$ by the definition of U. Therefore the first-order and envelope conditions are valid on U. The first-order condition is

$$(2-\lambda)\Big[K + c\Delta - 2c\,h(\Delta, y)\Big] - 2c\Big[1 + (2-\lambda)h(\Delta, y) + \lambda y\Big] + \delta W_0'\big(h(\Delta, y)\big) = 0, \tag{D.18}$$

and the envelope condition is

$$V_{0y}(y, \Delta) = \lambda\Big[K + c\Delta - 2c\,h(\Delta, y)\Big]. \tag{D.19}$$

The right-hand side of (D.19) is affine in (y, Δ), and V_{0y} is also affine on $(-\bar{\varepsilon}, \bar{\varepsilon}) \times (-\frac{1}{2}, \frac{1}{2})$. Since they agree on the nonempty open set U, they agree on the whole interior state space. Taking expectations over Δ' and using $E[\Delta'] = 0$, we obtain

$$W_0'(y) = \lambda(K - 2c\alpha) + 2\lambda c\rho\, y. \tag{D.20}$$

Substitute (D.20) and $h(\Delta, y) = \alpha + \beta\Delta - \rho y$ into (D.18). Matching the coefficient on y gives

$$\delta\lambda\rho^2 - 2(2-\lambda)\rho + \lambda = 0. \tag{D.21}$$

Moreover,

$$\rho = h\left(0, -\frac{1}{2}\right) - h\left(0, \frac{1}{2}\right),$$

and both terms lie in $[0, \frac{1}{2}]$, so $|\rho| \le \frac{1}{2} < 1$. Rearranging (D.21),

$$\lambda - (2-\lambda)\rho = \frac{\lambda}{2}(1 - \delta\rho^2) > 0. \tag{D.22}$$

Next consider firm 1's evaluation operator under the fixed profile (p^0, K):

$$(\mathcal{T}_1\Psi)(y, \Delta) = K\Big[(2-\lambda)\big(\tfrac{1}{2} - h(\Delta, y)\big) + \lambda\big(\tfrac{1}{2} - y\big)\Big] + \delta E\big[\Psi(h(\Delta, y), \Delta')\big].$$

Let $\mathcal{A}$ denote the space of affine functions in (y, Δ). Because current payoff is affine and h is affine, $\mathcal{T}_1$ maps $\mathcal{A}$ into itself. Since $\mathcal{T}_1$ is a contraction, its unique fixed point is affine:

$$V_1(y, \Delta) = A_1 + B_1 y + C_1\Delta.$$

Matching coefficients on y in the fixed-point equation $V_1 = \mathcal{T}_1 V_1$ gives

$$B_1 = K\big[(2-\lambda)\rho - \lambda\big] - \delta\rho B_1,$$

hence

$$B_1 = -\frac{K\big[\lambda - (2-\lambda)\rho\big]}{1 + \delta\rho} < 0, \tag{D.23}$$

where the inequality uses (D.22) and $1 + \delta\rho \ge 1 - \delta/2 > 0$.

Finally, evaluate firm 1's optimization problem at the state

$$(y, \Delta) = \left(\frac{1}{2}, 0\right).$$

Set

$$z_0 := h\left(0, \frac{1}{2}\right) = \frac{K - p^0(0, \frac{1}{2})}{2c} \in \left[0, \frac{1}{2}\right].$$

If firm 1 chooses a price $p \leq K$ at this state, then for z near z_0 with $z < z_0$ we may write

$$p = p^0\left(0, \frac{1}{2}\right) + 2cz,$$

and the next period's market shares are $y' = z$ (no truncation for such nearby z). Therefore firm 1's Bellman payoff from choosing z is

$$J_1(z) = \left[1 - (2-\lambda)z - \frac{\lambda}{2}\right]\left[p^0\left(0, \frac{1}{2}\right) + 2cz\right] + \delta\big(A_1 + B_1 z\big).$$

At $z = z_0$ this equals the equilibrium payoff, and since $p^0(0, \frac{1}{2}) + 2cz_0 = K$, we have

$$J_1'(z_0^-) = -(2-\lambda)K + 2c\left[1 - (2-\lambda)z_0 - \frac{\lambda}{2}\right] + \delta B_1.$$

But

$$1 - (2-\lambda)z_0 - \frac{\lambda}{2} = (2-\lambda)\left(\frac{1}{2} - z_0\right) \leq \frac{2-\lambda}{2},$$

so by (D.23),

$$J_1'(z_0^-) \leq (2-\lambda)(c-K) + \delta B_1 < 0.$$

Hence for some $z < z_0$ sufficiently close to z_0,

$$J_1(z) > J_1(z_0),$$

so firm 1 strictly prefers a price below K at the state $(\frac{1}{2}, 0)$. This contradicts $p^1 \equiv K$.

Therefore no affine MPE can satisfy $p^1 \equiv K$. By symmetry, no affine MPE can satisfy $p^i \equiv K$ for either firm. □

To complete the proof, we now combine the previous lemmas with the analysis of the interior case. Let (p^0, p^1) be any affine MPE. If $y_{t+1} = -\frac{1}{2}$ on a nonempty open set, then firm 0 would get no new consumers on that set, so its best response would be $p^0 = K$. Since p^0 is affine, this would imply $p^0 \equiv K$, contradicting Lemma 4. If $y_{t+1} = \frac{1}{2}$ on a nonempty open set, then firm 1 would get no new consumers on that set, so the same argument would imply $p^1 \equiv K$, again contradicting Lemma 4.

Therefore no affine MPE can have a nonempty open region where market shares are at the corner. Since the corner inequalities are affine (and therefore continuous), if one held strictly at a single state it would hold on a nonempty open neighborhood. Hence every affine MPE must lie on the interior branch everywhere, except possibly on lower-dimensional boundary sets. Similarly, if $p^i = K$ on a nonempty open set, affinity of prices would imply $p^i \equiv K$, impossible by Lemma 4. Thus the price cap is slack on any nonempty open set.

On that nonempty open set, the solutions to the problems of both firms are strictly interior and strictly below the cap, so the unconstrained Bellman first-order and envelope conditions from before are valid. The resulting identities are affine/polynomial in (Δ_t, y_t), so if they hold on a nonempty open set they hold everywhere. Hence the coefficient vector must satisfy equations (B1)–(B6). Solving that system gives

$$\mu = h = k = 0, \qquad \delta b^3 - 3b + \frac{2\lambda}{2-\lambda} = 0, \qquad a = \frac{2-\delta b^2}{2(3-\delta b^2)}, \qquad m = \frac{2+\delta b}{2-\lambda(1-\delta)}.$$

Therefore

$$p^0(\Delta_t, y_t) = c\big(m + a\Delta_t + by_t\big), \qquad p^1(\Delta_t, y_t) = c\big(m - a\Delta_t - by_t\big),$$

which is exactly the equilibrium in (12). This establishes that the MPE in which firms follow affine strategies is unique.

Proof of Proposition 2.

Using the market-share formula together with the equilibrium prices (12), gives

$$y_{t+1} = \frac{\Delta_t}{2} + \frac{p_R^1(y_t, \Delta_t) - p_R^0(y_t, \Delta_t)}{2c} = \left(\frac{1}{2} - a\right) \Delta_t - by_t,$$

which is a stable AR(1) because $b \in (0, 1)$.

Proof of Proposition 3.

From equation (6), the average price of a new policy without regulation equals:

$$\frac{p_U^0(\Delta_t) + p_U^1(\Delta_t)}{2} = c - \frac{\delta\lambda}{2-\lambda} K < c. \tag{D.24}$$

From equation (12), the average price with price walking regulation equals:

$$\frac{p_R^0(y_t, \Delta_t) + p_R^1(y_t, \Delta_t)}{2} = c\frac{2 + \delta b}{2 - \lambda(1-\delta)} > c. \tag{D.25}$$

Differentiating equation (D.24), we find that the average price without regulation is decreasing in λ. Let $b(\lambda)$ be the unique root of equation (8) in $(0, 1)$. By the implicit function theorem,

$$\frac{\partial b}{\partial \lambda}(\lambda) = \frac{4}{3(2-\lambda)^2(1-\delta b^2)} > 0,$$

so

$$\begin{aligned}\frac{d}{d\lambda}\left[\frac{p_R^0(y_t, \Delta_t) + p_R^1(y_t, \Delta_t)}{2}\right] &= c\frac{\delta}{2-\lambda(1-\delta)}\frac{\partial b}{\partial \lambda}(\lambda) + c\frac{2+\delta b}{[2-\lambda(1-\delta)]^2}(1-\delta) \\ &= \frac{c}{2-\lambda(1-\delta)}\left[\frac{4\delta}{3(2-\lambda)^2(1-\delta b^2)} + \frac{(2+\delta b)(1-\delta)}{2-\lambda(1-\delta)}\right] > 0,\end{aligned}$$

so the average price with regulation is increasing in λ. Therefore, the difference in average prices is increasing in the parameter λ.

Proof of Proposition 4.

Before presenting the proof, we obtain the expressions for the loss in total surplus with and without price walking regulation. As before, define the cutoff type as:

$$\ell^* = \frac{1+\Delta}{2} - \frac{p^0 - p^1}{2c} = \frac{1+\Delta}{2} + \frac{d}{2c}, \tag{D.26}$$

where $d := p^1 - p^0$. Total transportation costs among shoppers equal:

$$c\left[\int_0^{\ell^*} |\ell - \varepsilon_0|\, d\ell + \int_{\ell^*}^1 |\ell - (1-\varepsilon_1)|\, d\ell\right] = \frac{c}{2}\left[\varepsilon_0^2 + \varepsilon_1^2 + (\ell^* - \varepsilon_0)^2 + (1 - \varepsilon_1 - \ell^*)^2\right].$$

The efficient ("first-best") allocation minimizes transportation costs by setting the cutoff exactly in between both firms:

$$\ell^{FB} = \frac{\varepsilon_0 + 1 - \varepsilon_1}{2} = \frac{1+\Delta}{2}.$$

Substituting this expression in equation (D.26) gives $d = 0$. Transportation cost among active customers is minimized by setting the same price for both firms, so they buy from the nearest seller.

Define the loss of surplus associated with cutoff ℓ as:

$$\frac{c}{2}\left[\varepsilon_0^2+\varepsilon_1^2+(\ell-\varepsilon_0)^2+(1-\varepsilon_1-\ell)^2\right]-\frac{c}{2}\left[\varepsilon_0^2+\varepsilon_1^2+(\ell^{FB}-\varepsilon_0)^2+(1-\varepsilon_1-\ell^{FB})^2\right],$$

which, using $\ell^{FB} = \frac{1+\Delta}{2}$, can be expressed as

$$c \cdot \left(\ell - \ell^{FB}\right)^2. \tag{D.27}$$

Note that the surplus loss only depends on the demand shock through the first-best threshold ℓ^{FB}.

Using the cutoff among active customers (D.26) and the fact that there is a mass $2-\lambda$ of such customers, we obtain the active cohort's loss of surplus relative to the first-best:

$$(2-\lambda)c\left(\frac{1+\Delta_t}{2}+\frac{d_t}{2c}-\ell^{FB}\right)^2=(2-\lambda)c\left(\frac{1+\Delta_t}{2}+\frac{d_t}{2c}-\frac{1+\Delta_t}{2}\right)^2=\frac{2-\lambda}{4c}d_t^2.$$

Inertial consumers (mass λ) keep last period's assignment, so their cutoff at t remains at

$$\ell_{t-1}=\frac{1+\Delta_{t-1}}{2}+\frac{d_{t-1}}{2c}.$$

Their deviation from the first best ℓ_t^{FB} equals:

$$\ell_{t-1}-\ell_t^{FB}=\frac{\Delta_{t-1}-\Delta_t}{2}+\frac{d_{t-1}}{2c}.$$

Substituting in equation (D.27) and using the fact that there is a mass λ of inertial customers, we obtain their surplus loss in period t:

$$\lambda c\left(\frac{\Delta_{t-1}-\Delta_t}{2}+\frac{d_{t-1}}{2c}\right)^2.$$

Combining the surplus loss among active and inertial customers, we obtain the surplus loss among all consumers in period t conditional on the demand shocks:

$$\frac{2-\lambda}{4c}d_t^2+\lambda c\left(\frac{\Delta_{t-1}-\Delta_t}{2}+\frac{d_{t-1}}{2c}\right)^2. \tag{D.28}$$

We now use expression (D.28) to evaluate the surplus loss in each case.

No Regulation

Without regulation, active customers face prices (6), so the price difference equals:

$$d_t = -\frac{2}{3}c\Delta_t \quad \forall t.$$

Substituting in (D.28), gives the surplus loss conditional on the realized demand shocks:

$$\frac{2-\lambda}{9}c\Delta_t^2+\lambda c\left(\frac{\Delta_{t-1}}{6}-\frac{\Delta_t}{2}\right)^2$$

Since net shocks are symmetric around zero, we have $E[\Delta_t] = 0$ for all t. Denote the variance of each period's demand shock by $\sigma^2 := Var(\Delta_t) = E[\Delta_t^2]$ and recall that $E[\Delta_t \Delta_{t-1}] = 0$ since shocks are i.i.d. with zero mean. Taking expectations, we obtain the expected surplus loss:

$$\frac{2-\lambda}{9} c\sigma^2 + \lambda c\sigma^2 \left(\frac{1}{36} + \frac{1}{4} \right) = \frac{4+3\lambda}{18} c\sigma^2. \tag{D.29}$$

Regulation

Propositions 1 and 2 give

$$d_t = -2c\left(a\Delta_t + by_t\right), \quad y_{t+1} = -by_t + \left(\frac{1}{2} - a\right)\Delta_t.$$

Note that y_t is independent of Δ_t. The unconditional mean and variance of y_t are

$$E(y_t) = 0, \qquad Var(y_t) = \frac{\left(\frac{1}{2} - a\right)^2}{1-b^2}\sigma^2.$$

The variance of the price difference equals

$$E[d_t^2] = E\left[\left(-2c\left(a\Delta_t + by_t\right)\right)^2\right] = 4c^2\sigma^2 \left[a^2 + \frac{b^2}{1-b^2}\left(\frac{1}{2} - a\right)^2 \right]. \tag{D.30}$$

where we used the expression for the unconditional variance of y_t and $E[\Delta_t y_t] = 0$.

For the inertial customers term, note that

$$\frac{\Delta_{t-1} - \Delta_t}{2} + \frac{d_{t-1}}{2c} = \left(\frac{1}{2} - a\right)\Delta_{t-1} - by_{t-1} - \frac{1}{2}\Delta_t.$$

Because Δ_t is independent of (Δ_{t-1}, y_{t-1}) and Δ_{t-1} is independent of y_{t-1}, we have

$$E\left[\left(\frac{\Delta_{t-1} - \Delta_t}{2} + \frac{d_{t-1}}{2c}\right)^2\right] = \left[\frac{1}{4} + \frac{\left(\frac{1}{2} - a\right)^2}{1-b^2}\right]\sigma^2. \tag{D.31}$$

Substituting (D.30) and (D.31) in (D.28), gives the expected surplus loss with regulation:

$$c\sigma^2 \left\{ (2-\lambda)a^2 + \frac{\left[(2-\lambda)b^2 + \lambda\right]\left(\frac{1}{2} - a\right)^2}{1-b^2} + \frac{\lambda}{4} \right\}. \tag{D.32}$$

Surplus Comparison

Comparing the expressions for surplus losses in (D.29) and (D.32), total surplus is higher with regulation if and only if:

$$\frac{4+3\lambda}{18} \geq (2-\lambda)\, a^2 + \frac{\left[(2-\lambda)\, b^2 + \lambda\right]\left(\frac{1}{2} - a\right)^2}{1-b^2} + \frac{\lambda}{4}. \tag{D.33}$$

Let

$$G(\delta, \lambda) := \frac{4+3\lambda}{18} - \left\{ (2-\lambda)\, a_{\delta,\lambda}^2 + \frac{\left[(2-\lambda)\, b_{\delta,\lambda}^2 + \lambda\right]\left(\frac{1}{2} - a_{\delta,\lambda}\right)^2}{1-b_{\delta,\lambda}^2} + \frac{\lambda}{4} \right\},$$

where we write $a_{\delta,\lambda}$ and $b_{\delta,\lambda}$ to emphasize that these terms are functions of δ and λ. Substituting $a_{0,\lambda} = \frac{1}{3}$ and $b_{0,\lambda} = \frac{2\lambda}{3(2-\lambda)}$ and simplifying, gives

$$G(0,\lambda) = -\frac{2\lambda^2}{9\,(6-5\lambda)\,(6-\lambda)} < 0,$$

so regulation lowers surplus for δ low enough. Computing the derivatives and applying the implicit function theorem, one finds that the solution $\lambda^*(\delta)$ is strictly increasing. Moreover, $\lambda^*(\frac{3}{4}) = 0$ (so, by strict monotonicity, there are no positive λ satisfying (D.33) for $\delta \leq \frac{3}{4}$) and $\lambda^*(1) \approx 0.3109966$.

Proof of Proposition 5.

We first compute each firm's expected per-period profit with and without regulation.

Without regulation, equations (6) and(7) imply

$$s_t = \frac{1}{2} + \frac{\Delta_t}{6}, \qquad p_0^U(\Delta_t) = c - \frac{\delta\lambda}{2-\lambda}K + \frac{c}{3}\Delta_t.$$

Hence firm 0's period-t profit is

$$\Pi_{t,0}^U = (2-\lambda)s_t\, p_0^U(\Delta_t) + \lambda s_{t-1}K.$$

Using $E[\Delta_t] = 0$, $E[\Delta_t^2] =: \sigma^2$, and $E[s_{t-1}] = 1/2$, we obtain

$$\bar{\pi}_U(\delta,\lambda,K) = E[\Pi_{t,0}^U] = \frac{\lambda(1-\delta)}{2}K + \frac{2-\lambda}{2}\left(1 + \frac{\sigma^2}{9}\right)c.$$

Under regulation, Proposition 2 implies that normalized market shares follow a stationary process, with unconditional moments

$$E[y_t] = 0, \quad Var(y_t) = \frac{\left(\frac{1}{2}-a\right)^2}{1-b^2}\sigma^2.$$

Moreover, Δ_t is independent of y_t and

$$s_t = \frac{1}{2} + y_{t+1} = \frac{1}{2} - by_t + \left(\frac{1}{2} - a\right)\Delta_t.$$

From Proposition 1,

$$p_0^R(y_t,\Delta_t) = c(m + a\Delta_t + by_t).$$

Therefore

$$\begin{aligned}\Pi_{t,0}^R &= \left[(2-\lambda)s_t + \lambda\left(\tfrac{1}{2} + y_t\right)\right]p_0^R(y_t,\Delta_t)\\ &= c\left\{1 + (2-\lambda)\left(\tfrac{1}{2} - a\right)\Delta_t + \left[\lambda - (2-\lambda)b\right]y_t\right\}(m + a\Delta_t + by_t).\end{aligned}$$

Taking expectations and using the unconditional moments above gives

$$\bar{\pi}_R(\delta,\lambda) = cm + c\sigma^2\left[(2-\lambda)a\left(\tfrac{1}{2} - a\right) + \left[\lambda - (2-\lambda)b\right]b\frac{\left(\frac{1}{2}-a\right)^2}{1-b^2}\right].$$

We now prove the two claims. For the first claim, fix (δ,λ). The expression for $\bar{\pi}_U(\delta,\lambda,K)$ is affine in K with strictly positive slope $\lambda(1-\delta)/2$, whereas $\bar{\pi}_R(\delta,\lambda)$ does not depend on K. Hence there exists a unique cutoff $\bar{K}(\delta, \lambda)$ such that

$$\bar{\pi}_U(\delta, \lambda, K) \gtrless \bar{\pi}_R(\delta, \lambda) \quad \iff \quad K \gtrless \bar{K}(\delta, \lambda).$$

It remains to show that $\bar{K}(\delta, \lambda) > 0$.

At $K = 0$,

$$\bar{\pi}_R - \bar{\pi}_U\big|_{K=0} = c\left[m - \frac{2-\lambda}{2} + \sigma^2 \Gamma(\delta, \lambda)\right],$$

where

$$\Gamma(\delta, \lambda) := (2-\lambda)a\left(\tfrac{1}{2} - a\right) + \left[\lambda - (2-\lambda)b\right]b\frac{\left(\frac{1}{2} - a\right)^2}{1 - b^2} - \frac{2-\lambda}{18}.$$

The first term is strictly positive because $m \in (1, 2)$ and $(2-\lambda)/2 < 1$.

For the second term, use

$$a = \frac{2 - \delta b^2}{2(3 - \delta b^2)} \qquad \text{and} \qquad 2\lambda = (2-\lambda)b(3 - \delta b^2),$$

where the second equality is a rearrangement of equation (8). Algebraic manipulations give

$$\Gamma(\delta, \lambda) = \frac{(2-\lambda)b^2\, h(b^2)}{72(1-b^2)(3-\delta b^2)^2},$$

where $h(z) := 4\delta^2 z^2 - (4\delta^2 + 15\delta)z + 6\delta + 9$. Differentiating, gives

$$h'(z) = 8\delta^2 z - 4\delta^2 - 15\delta < 0 \quad \forall z \in [0, 1],$$

so h is strictly decreasing on $[0, 1]$. Because $b < 1$, we have $b^2 < 1$, and therefore

$$h(b^2) > h(1) = 9(1 - \delta) > 0.$$

Hence $\Gamma(\delta, \lambda) > 0$, so $\bar{\pi}_R > \bar{\pi}_U|_{K=0}$, establishing that $\bar{K}(\delta, \lambda) > 0$.

For the second claim, fix (K, λ) and define

$$\Delta_{K,\lambda}(\delta) := \bar{\pi}_R(\delta, \lambda) - \bar{\pi}_U(\delta, \lambda, K).$$

By continuity at $\delta = 1$, it suffices to show that $\lim_{\delta \uparrow 1} \Delta_{K,\lambda}(\delta) > 0$.

Let $b_\delta \in (0, 1)$ denote the unique root of

$$\delta b^3 - 3b + \frac{2\lambda}{2-\lambda} = 0$$

(we omit the dependence of b_δ on $\lambda \in (0, 1)$ to simplify notation). Because

$$\frac{\partial}{\partial b}\left(\delta b^3 - 3b + \frac{2\lambda}{2-\lambda}\right) = 3\delta b^2 - 3 < 0 \qquad \text{for } (\delta, b) \in (0, 1] \times (0, 1),$$

the root b_δ is unique and depends continuously on δ. Hence b_δ converges to $b_1 \in (0, 1)$ as $\delta \uparrow 1$, where b_1 solves

$$b_1^3 - 3b_1 + \frac{2\lambda}{2-\lambda} = 0 \therefore 2\lambda = (2-\lambda)\, b_1 \left(3 - b_1^2\right). \tag{D.34}$$

Define

$$a_\delta := \frac{2 - \delta b_\delta^2}{2(3 - \delta b_\delta^2)}, \qquad m_\delta := \frac{2 + \delta b_\delta}{2 - \lambda(1-\delta)},$$

and note that a_δ converges to $a_1 := \frac{2-b_1^2}{2(3-b_1^2)}$ and m_δ converges to $1+\frac{b_1}{2}$ as $\delta \uparrow 1$.

Using the expression for $\bar{\pi}_U$,

$$\lim_{\delta\uparrow 1} \bar{\pi}_U(\delta,\lambda,K) = \frac{2-\lambda}{2}\left(1+\frac{\sigma^2}{9}\right)c.$$

Using the expression for $\bar{\pi}_R$, continuity, and the equations for a_1 and b_1 above, we obtain

$$\lim_{\delta\uparrow 1} \bar{\pi}_R(\delta,\lambda) = c\left(1+\frac{b_1}{2}\right) + c\sigma^2(2-\lambda)\frac{4-b_1^2}{8(3-b_1^2)^2}.$$

Subtracting these expressions, gives

$$\lim_{\delta\uparrow 1} \Delta_{K,\lambda}(\delta) = c\left[\frac{\lambda+b_1}{2} + \sigma^2(2-\lambda)\frac{b_1^2(15-4b_1^2)}{72(3-b_1^2)^2}\right].$$

This expression is strictly positive because $\lambda \in (0,1)$, $b_1 \in (0,1)$, and $15-4b_1^2>0$. Hence, $\lim_{\delta\uparrow 1}\Delta_{K,\lambda}(\delta)>0$, and continuity yields a cutoff $\bar{\delta}(K,\lambda)<1$ such that

$$\bar{\pi}_R(\delta,\lambda) > \bar{\pi}_U(\delta,\lambda,K) \qquad \text{for all } \delta > \bar{\delta}(K,\lambda).$$

This completes the proof.

Proof of Proposition 6.

Existence

From $T+1$ onward, the game is exactly as in the regulated model. As shown in Lemma 1, the value function for all such periods is:

$$V_0(y_t,\Delta_t) = A + By + C\Delta + Dy^2 + Ey\Delta + F\Delta^2,$$

and therefore

$$W_0(y) := E_t\left[V_0(y,\Delta_{t+1})\right] = A + By + Dy^2 + F\sigma^2,$$

Differentiating, gives

$$W_0'(y) = B + 2Dy.$$

At date T, firm 0 solves the same problem as in the unregulated model except the continuation value function is the one from the regulated model:

$$V_0^A(y_T,\Delta_T) = \begin{array}{l} \max_{p_t^0, y_{T+1}} \left\{(2-\lambda)\left(y_{T+1}+\frac{1}{2}\right)p_T^0 + \lambda\left(y_T+\frac{1}{2}\right)K + \delta W_0(y_{T+1})\right\} \\ \text{subject to } y_{T+1} = \frac{\Delta_T}{2} + \frac{c(n-a\Delta_T)-p_T^0}{2c} \end{array}.$$

Substituting the constraint in the objective function, gives

$$V_0^A(y_T,\Delta_T) = \max_{y_{T+1}}\left\{c(2-\lambda)\left(y_{T+1}+\frac{1}{2}\right)\left[n+\Delta_T(1-a)-2y_{T+1}\right] + \lambda\left(y_T+\frac{1}{2}\right)K + \delta W_0(y_{T+1})\right\}. \tag{D.35}$$

Moreover, the constraint gives

$$p_T^0 = c(n+a\Delta_T) \iff y_{T+1} = \left(\frac{1}{2}-a\right)\Delta_T.$$

The objective function is a strictly concave, quadratic function and the necessary and sufficient FOC for an interior optimum is:

$$c(2-\lambda)\left[n+\Delta_T\left(1-a\right)-2y_{T+1}\right]-2c(2-\lambda)\left(y_{T+1}+\frac{1}{2}\right)+\delta W_0'(y_{T+1})=0.$$

Evaluating the LHS at $y_{T+1}=\left(\frac{1}{2}-a\right)\Delta_T$ and using the expression for W_0', gives

$$c(2-\lambda)\left[n+\Delta_T\left(1-a\right)-2\left(\frac{1}{2}-a\right)\Delta_T\right]-2c(2-\lambda)\left(\left(\frac{1}{2}-a\right)\Delta_T+\frac{1}{2}\right)+\delta\left[B+2D\left(\frac{1}{2}-a\right)\Delta_T\right]$$

$$=c(2-\lambda)\left[n-1+\left(3a-1\right)\Delta_T\right]+\delta\left[B+2D\left(\frac{1}{2}-a\right)\Delta_T\right].$$

Using the definitions of m, B, and b, we can write $B=c(\lambda m-b)$. Substituting this condition along with $D=\frac{c(2-\lambda)b^2}{2}$, and $n=1-\frac{\delta b(2-\delta b^2)}{2-\lambda(1-\delta)}$, the expression above becomes

$$c(2-\lambda)\left[\left(3a-1\right)\Delta_T-\frac{\delta b(2-\delta b^2)}{2-\lambda(1-\delta)}\right]+\delta c\left[\lambda m-b+(2-\lambda)b^2\left(\frac{1}{2}-a\right)\Delta_T\right]$$

$$=c(2-\lambda)\left[3a-1+\delta b^2\left(\frac{1}{2}-a\right)\right]\Delta_T+\delta c\left[(\lambda m-b)-(2-\lambda)\frac{b(2-\delta b^2)}{2-\lambda(1-\delta)}\right]$$

We claim that this expression equals zero for all Δ_T. To see this, note that substituting $a:=\frac{2-\delta b^2}{2(3-\delta b^2)}$ establishes that

$$3a-1+\delta b^2\left(\frac{1}{2}-a\right)=0.$$

For the term inside the second brackets, note that substituting $m=\frac{2+\delta b}{2-\lambda(1-\delta)}$ and rearranging gives

$$\frac{\lambda\left(2+\delta b\right)-b(2-\lambda)(2-\delta b^2)}{2-\lambda(1-\delta)}-b=0\iff\delta b^3-3b+\frac{2\lambda}{2-\lambda}=0,$$

which is true by the definition of b. Therefore, $p_T^0=c(n+a\Delta_T)$ solves firm 0's problem at date T. A symmetric argument establishes optimality of $p_T^1=c(n-a\Delta_T)$ for firm 1.

It remains to verify that these candidate prices are feasible and induce interior market shares. First, as in Proposition 1,

$$|y_{T+1}|=\left(\frac{1}{2}-a\right)|\Delta_T|\leq\left(\frac{1}{2}-a\right)\bar{\varepsilon}<\frac{1}{2},$$

where the strict inequality follows from condition (10) and the definition of a. Thus the market-share truncation also does not bind at date T.

Second, the price cap is slack. Using the definitions of m and n,

$$m-n=\frac{2+\delta b}{2-\lambda(1-\delta)}-\left[1-\frac{\delta\left(2b-\delta b^3\right)}{2-\lambda(1-\delta)}\right].$$

Since $2b-\delta b^3=\frac{2\lambda}{2-\lambda}-b$ by equation (8), we have

$$m-n=\frac{2+\delta b}{2-\lambda(1-\delta)}-\left[1-\frac{\delta}{2-\lambda(1-\delta)}\left(\frac{2\lambda}{2-\lambda}-b\right)\right]$$

$$= \frac{1}{2-\lambda(1-\delta)}\left[\frac{2\lambda\delta}{2-\lambda} + \lambda(1-\delta)\right] > 0.$$

Therefore,

$$p_T^i \le c(n + a\bar{\varepsilon}) < c(m + a\bar{\varepsilon} + b/2) \le K \qquad (i = 0, 1),$$

where the last inequality is the lower-bound restriction on K from (11).

Finally, we verify that there are no profitable deviations to off-path market shares where the truncation binds. As shown in Proposition 1, it suffices to verify that firm 0 does not profit by setting $p_T^0 = K$ and not selling to any new customer ($y_{T+1} = -1/2$). Let

$$p_T^-(\Delta_T) := c(n + 1 + (1-a)\Delta_T),$$

so that $y_{T+1} = -1/2$ if and only if $p_0^T \ge p_T^-(\Delta_T)$.

There are two cases. If $K < p_T^-(\Delta_T)$, then no feasible price can induce $y_{T+1} = -1/2$, so this deviation is not possible.

If $K \ge p_T^-(\Delta_T)$, then every $p_T^0 \in [p_T^-(\Delta_T),\ K]$ induces $y_{T+1} = -1/2$. On this region, the firm attracts no new customers. Since renewal prices at date T are still set at K, the payoff from any such deviation is the same:

$$\lambda\left(y_T + \frac{1}{2}\right)K + \delta W_0\left(-\frac{1}{2}\right).$$

But setting $p_T^0 = p_T^-(\Delta_T)$ is equivalent to choosing $y_{T+1} = -1/2$ in program (D.35). This is feasible but not optimal since the unique solution to that program is interior. Hence, this is also not a profitable deviation. Intuitively, there is no gain from selling to no new customers, when prices of passive customers can be set separately.

Uniqueness

From date $T+1$ onward, the game coincides with the regulated game. Hence, by Proposition 1, the continuation values satisfy

$$W_0(y) = \bar{A} + By + Dy^2, \qquad W_1(y) = \bar{A} - By + Dy^2,$$

where $\bar{A} := A + F\sigma^2$, $B = c(\lambda m - b)$, and $D = \frac{c(2-\lambda)b^2}{2}$. Therefore,

$$W_0'(y) = c(\lambda m - b) + c(2-\lambda)b^2 y, \qquad W_1'(y) = -c(\lambda m - b) + c(2-\lambda)b^2 y.$$

Since renewal prices at date T are unconstrained by the regulation, each firm optimally sets

$$r_{0T} = r_{1T} = K.$$

Thus uniqueness at date T reduces to uniqueness of prices offered to new customers.

Suppose that there exists an equilibrium in affine strategies at date T, and write

$$p_{0T}(y_T, \Delta_T) = c(\alpha_0 + \beta_0\Delta_T + \gamma_0 y_T), \qquad p_{1T}(y_T, \Delta_T) = c(\alpha_1 + \beta_1\Delta_T + \gamma_1 y_T).$$

Define the non-truncated next-period market share by

$$\widetilde{y}_{T+1} := \frac{\Delta_T}{2} + \frac{p_{1T}(y_T, \Delta_T) - p_{0T}(y_T, \Delta_T)}{2c}.$$

The actual next-period market share is the truncation of $\widetilde{y}_{T+1}$ to $[-1/2, 1/2]$.

By the same boundary-deviation arguments used in the existence part of Proposition 1, an affine equilibrium at date T cannot bind either the price cap or the truncation on a nonempty open set of states. Hence there exists a nonempty open set $U \subset (-1/2, 1/2) \times (-\bar{\varepsilon}, \bar{\varepsilon})$ such that, for every $(y_T, \Delta_T) \in U$,

$$p_{0T}(y_T, \Delta_T) < K, \qquad p_{1T}(y_T, \Delta_T) < K, \qquad y_{T+1} = \widetilde{y}_{T+1} \in (-1/2, 1/2).$$

On U, both firms' optimization problems are interior, and the first-order conditions are necessary and sufficient.

For firm 0, the objective function of the date-T Bellman on U is

$$(2-\lambda)\Big(\frac{1}{2} + y_{T+1}\Big)p_{0T} + \lambda\Big(\frac{1}{2} + y_T\Big)K + \delta W_0(y_{T+1}),$$

with $\frac{\partial y_{T+1}}{\partial p_{0T}} = -\frac{1}{2c}$, so the first-order condition is

$$(2-\lambda)\Big(\frac{1}{2} + y_{T+1}\Big) - \frac{2-\lambda}{2c}p_{0T} - \frac{\delta}{2c}W_0'(y_{T+1}) = 0. \tag{D.36}$$

Similarly, for firm 1, the objective is

$$(2-\lambda)\Big(\frac{1}{2} - y_{T+1}\Big)p_{1T} + \lambda\Big(\frac{1}{2} - y_T\Big)K + \delta W_1(y_{T+1}),$$

with $\frac{\partial y_{T+1}}{\partial p_{1T}} = \frac{1}{2c}$, so the first-order condition is

$$(2-\lambda)\Big(\frac{1}{2} - y_{T+1}\Big) - \frac{2-\lambda}{2c}p_{1T} + \frac{\delta}{2c}W_1'(y_{T+1}) = 0. \tag{D.37}$$

Substitute the expressions for W_0' and W_1' into (D.36)–(D.37) and use the definition of n to obtain

$$p_{0T}(y_T, \Delta_T) = c\left[n + (2 - \delta b^2)y_{T+1}\right], \tag{D.38}$$

$$p_{1T}(y_T, \Delta_T) = c\left[n - (2 - \delta b^2)y_{T+1}\right]. \tag{D.39}$$

Because the strategies are affine, on U we have

$$y_{T+1} = \frac{\alpha_1 - \alpha_0}{2} + \frac{1 + \beta_1 - \beta_0}{2}\Delta_T + \frac{\gamma_1 - \gamma_0}{2}y_T.$$

Substituting this expression together with the affine forms for p_{0T} and p_{1T} into (D.38)–(D.39), we obtain two affine identities in (y_T, Δ_T) that hold on the nonempty open set U. Therefore they hold on the whole state space. Equating coefficients yields a linear system for $(\alpha_i, \beta_i, \gamma_i)$.

Adding (D.38) and (D.39) gives

$$p_{0T}(y_T, \Delta_T) + p_{1T}(y_T, \Delta_T) = 2cn,$$

hence

$$\alpha_0 + \alpha_1 = 2n, \qquad \beta_0 + \beta_1 = 0, \qquad \gamma_0 + \gamma_1 = 0.$$

Subtracting (D.38) from (D.39) gives

$$p_{1T}(y_T, \Delta_T) - p_{0T}(y_T, \Delta_T) = -2c(2 - \delta b^2)y_{T+1}.$$

Using the law of motion for y_{T+1}, we obtain

$$p_{1T} - p_{0T} = -c(2-\delta b^2)\left[(\alpha_1-\alpha_0) + (1+\beta_1-\beta_0)\Delta_T + (\gamma_1-\gamma_0)y_T\right].$$

Equating coefficients and using $\beta_1 = -\beta_0$ and $\gamma_1 = -\gamma_0$, we obtain

$$\alpha_1 - \alpha_0 = -(2-\delta b^2)(\alpha_1-\alpha_0),$$

$$\beta_1 - \beta_0 = -(2-\delta b^2)(1+\beta_1-\beta_0),$$

$$\gamma_1 - \gamma_0 = -(2-\delta b^2)(\gamma_1-\gamma_0).$$

Since $2-\delta b^2 > 0$, the first and third equations imply $\alpha_1 = \alpha_0$ and $\gamma_1 = \gamma_0$. Combined with $\alpha_0 + \alpha_1 = 2n$ and $\gamma_0 + \gamma_1 = 0$, this yields $\alpha_0 = \alpha_1 = n$ and $\gamma_0 = \gamma_1 = 0$.

For the slope coefficients, using $\beta_1 = -\beta_0$, the second equation becomes

$$2\beta_0 = (2-\delta b^2)(1-2\beta_0).$$

Therefore $\beta_0 = \frac{2-\delta b^2}{2(3-\delta b^2)} = a$ and $\beta_1 = -a$.

Thus any affine equilibrium at date T must satisfy

$$p_{0T}(y_T,\Delta_T) = c(n + a\Delta_T), \qquad p_{1T}(y_T,\Delta_T) = c(n - a\Delta_T).$$

Since the existence proof already shows that this affine strategy profile is an MPE, it follows that the MPE in affine strategies is unique.

Proof of Corollary 1.

The price increase at $T+1$ equals

$$p_R^{T+1} - p_R^T = c(m-n) = c\frac{\lambda(1-\delta) + \delta\left(3b - \delta b^3\right)}{2-\lambda(1-\delta)},$$

where we used the expressions for m and n. Substituting $3b - \delta b^3 = \frac{2\lambda}{2-\lambda}$ from equation (8) and rearranging, gives

$$p_R^{T+1} - p_R^T = c\frac{\lambda}{2-\lambda},$$

which is positive and increasing in $\lambda \in (0,1)$.

Next, consider the price increase at T:

$$\Delta p(\lambda;K) := \bar{p}_T^R - \bar{p}^U = \frac{\delta\lambda}{2-\lambda}K - c\,\frac{\delta b(\lambda)\left(2-\delta b(\lambda)^2\right)}{2-\lambda(1-\delta)}, \tag{D.40}$$

where $b(\lambda)$ is the unique solution to (8) in $(0,1)$. We need to show that $\Delta p > 0$ and $\frac{\partial \Delta p}{\partial \lambda} > 0$ for all $K \in [\underline{K}, \bar{K}]$.

Using equation (8), gives

$$\Delta p(\lambda;K) = \frac{\delta b\left(3-\delta b^2\right)}{2}\left\{K - c\frac{2\left(2-\delta b^2\right)}{\left[2-\lambda(1-\delta)\right]\left(3-\delta b^2\right)}\right\} > 0,$$

where the inequality holds because the expression inside brackets is strictly positive by equation (11).

We now show that $\frac{\partial \Delta p}{\partial \lambda} > 0$ for each fixed (λ, K). Differentiating equation (8) with respect to λ gives

$$b'(\lambda) = \frac{4}{3(2-\lambda)^2(1-\delta b^2)} > 0, \tag{D.41}$$

and, by equation (D.40),

$$\frac{\partial \Delta p}{\partial \lambda} = \frac{2\delta}{(2-\lambda)^2} K - c\delta \left[\frac{(2-3\delta b^2)b'(\lambda)}{2-\lambda(1-\delta)} + \frac{(1-\delta)b(2-\delta b^2)}{[2-\lambda(1-\delta)]^2} \right]. \tag{D.42}$$

For fixed λ, this is an increasing function of K with slope $\frac{2\delta}{(2-\lambda)^2} > 0$. Equation (11) gives the lower bound:

$$K \geq \underline{K} = c\left(m + a\bar{\epsilon} + \frac{b}{2}\right) > c\left(m + \frac{b}{2}\right) =: K_0,$$

where the inequality follows from $a > 0$ and $\bar{\epsilon} > 0$. Therefore, it suffices to verify that $\frac{\partial \Delta p}{\partial \lambda}(\lambda; K_0) > 0$.

Taking $x := \delta b^2 \in (0,1)$, we can rewrite (8) as

$$\lambda = \frac{2b(3-x)}{2+b(3-x)}. \tag{D.43}$$

Substituting (D.41) and (D.43) into (D.42), evaluating at $K = K_0$, and simplifying, yields

$$\left.\frac{\partial \Delta p}{\partial \lambda}\right|_{K=K_0} = \frac{c\,x\,(2+3b-bx)^2}{24\,b^2(1-x)(2b+3x-x^2)^2} \cdot Z(b,x), \tag{D.44}$$

where

$$\begin{aligned} Z(b,x) = {} & 2b^3x(5-3x) + b^2\big(10x^3 - 42x^2 + 36x + 4\big) \\ & + bx(3-x)\big(3x^3 - 12x^2 + 3x + 8\big) + 3x^2(1-x)(3-x). \end{aligned}$$

Since the term multiplying $Z(b,x)$ in (D.44) is strictly positive, it remains to show that $Z(b,x) > 0$ for all $(b,x) \in (0,1)^2$.

Let $r(x) := 10x^3 - 42x^2 + 36x + 4$ and $q(x) := 3x^3 - 12x^2 + 3x + 8$. Then

$$Z(b,x) = 2b^3x(5-3x) + b^2r(x) + bx(3-x)q(x) + 3x^2(1-x)(3-x). \tag{D.45}$$

Since $b, x \in (0,1)$, the first and last terms are strictly positive.

Note that r is strictly concave on $[0,1]$ since $r''(x) = 60x - 84 < 0$. By concavity,

$$r(x) \geq (1-x)r(0) + xr(1) = (1-x)\cdot 4 + x \cdot 8 = 4 + 4x > 0.$$

Similarly, q is strictly concave on $[0,1]$ since $q''(x) = 18x - 24 < 0$. By concavity,

$$q(x) \geq (1-x)q(0) + xq(1) = (1-x)\cdot 8 + x \cdot 2 = 8 - 6x > 0.$$

Hence, the second and third terms are also strictly positive. Since each term in (D.45) is strictly positive, $Z(b,x) > 0$ for all $b, x \in (0,1)$. By (D.44), this implies

$$\frac{\partial \Delta p}{\partial \lambda}(\lambda, K) \geq \frac{\partial \Delta p}{\partial \lambda}(\lambda, K_0) > 0 \qquad \forall K \geq \underline{K}.$$

E Equilibria in Polynomial Strategies

This appendix extends the analysis in the text to arbitrary *interior symmetric MPE in polynomial strategies.* That is, we focus on MPE in which

- p_0 and p_1 are bivariate polynomials in (y, Δ);
- $p_1(y, \Delta) = p_0(-y, -\Delta)$ for all (y, Δ); and
- neither the truncation of market shares nor the price cap bind.

To simplify the expressions, we normalize prices by transportation costs $c > 0$:

$$P_i(y, \Delta) := \frac{p_i(y, \Delta)}{c}, \qquad i \in \{0, 1\}.$$

By symmetry,

$$P_1(y, \Delta) = P_0(-y, -\Delta). \tag{E.1}$$

Since we focus on interior equilibria, market shares evolve according to $y_{t+1} = z(y_t, \Delta_t)$, where

$$z(y, \Delta) := \frac{\Delta + P_1(y, \Delta) - P_0(y, \Delta)}{2}. \tag{E.2}$$

As in the proof of Proposition 1, firm 0's Bellman equation is

$$\begin{aligned} V_0(y_t, \Delta_t) = \quad & \max\nolimits_{y_{t+1}, p_t^0} \Big[(2-\lambda)\left(y_{t+1} + \tfrac{1}{2}\right) + \lambda\left(y_t + \tfrac{1}{2}\right)\Big] p_t^0 + \delta E_t\big[V_0(y_{t+1}, \Delta_{t+1})\big] \\ & \text{subject to } y_{t+1} = \tfrac{\Delta_t}{2} + \tfrac{p_t^1(y_t, \Delta_t) - p_t^0}{2c} \end{aligned}$$

Letting $v_i(y, \Delta) := \frac{V_i(y, \Delta)}{c}$ denote normalized profits and using the constraint to eliminate P_0 from the objective, the Bellman equation becomes

$$v_0(y, \Delta) = \max_z \Big[1 + \lambda y + z(2-\lambda)\Big] (\Delta + P_1(y, \Delta) - 2z) + \delta E\big[v_0(z, \Delta')\big].$$

The necessary first-order condition for an interior optimum is

$$(2-\lambda)\underbrace{(\Delta + P_1(y, \Delta) - 2z)}_{P_0(y, \Delta)} - 2\Big[1 + \lambda y + z(2-\lambda)\Big] + \delta E\left[\frac{\partial v_0}{\partial y}(z, \Delta')\right] = 0,$$

which can be rearranged as

$$1 + \lambda y + (2-\lambda)z - \frac{2-\lambda}{2} P_0(y, \Delta) - \frac{\delta}{2} E\left[\frac{\partial v_0}{\partial y}(z, \Delta')\right] = 0. \tag{E.3}$$

By the envelope theorem,

$$\begin{aligned} \frac{\partial v_0}{\partial y}(y, \Delta) &= \lambda(\Delta + P_1(y, \Delta) - 2z) + \Big[1 + \lambda y + (2-\lambda)z\Big] \frac{\partial P_1}{\partial y}(y, \Delta) \\ &= \lambda P_0(y, \Delta) + \Big[1 + \lambda y + (2-\lambda)z\Big] \frac{\partial P_1}{\partial y}(y, \Delta). \end{aligned}$$

Evaluate at next period's state (z, Δ') and take expectations:

$$E\left[\frac{\partial v_0}{\partial y}(z,\Delta')\right] = E\left\{\lambda P_0(z,\Delta') + \Big[1+\lambda z + (2-\lambda)z'\Big]\frac{\partial P_1}{\partial y}(z,\Delta')\right\},$$

where

$$z' = \frac{\Delta' + P_1(z,\Delta') - P_0(z,\Delta')}{2}. \tag{E.4}$$

Substituting back into equation (E.3), gives firm 0's Euler equation:

$$\begin{aligned} 0 = \quad & 1+\lambda y + (2-\lambda)z - \tfrac{2-\lambda}{2}P_0(y,\Delta) \\ & -\tfrac{\delta}{2}E\left\{\lambda P_0(z,\Delta') + \Big[1+\lambda z+(2-\lambda)z'\Big]\tfrac{\partial P_1}{\partial y}(z,\Delta')\right\} \end{aligned}, \tag{E.5}$$

where z' is given by (E.4). A symmetric argument gives firm 1's Euler equation:

$$\begin{aligned} 0 &= 1 - \lambda y - (2-\lambda)z - \frac{2-\lambda}{2}P_1(y,\Delta) \\ &\quad + \frac{\delta}{2}E\Big\{-\lambda P_1(z,\Delta') + \big(1-\lambda z - (2-\lambda)z'\big)\frac{\partial P_0}{\partial y}(z,\Delta')\Big\}. \end{aligned} \tag{E.6}$$

Define the average price A and the price gap B in equilibrium:

$$A(y,\Delta) := \frac{P_0(y,\Delta)+P_1(y,\Delta)}{2}, \qquad B(y,\Delta) := \frac{P_0(y,\Delta)-P_1(y,\Delta)}{2},$$

so $P_0 = A + B$ and $P_1 = A - B$.

It is convenient to work with A and B rather than P_0 and P_1. By equation (E.1) (symmetry), A is an even function and B is an odd function under the transformation $(y,\Delta) \mapsto (-y,-\Delta)$. Moreover, the law of motion for market shares can be written as

$$z(y,\Delta) = \frac{\Delta}{2} - B(y,\Delta).$$

Adding and subtracting (E.5)-(E.6) gives the equivalent system:

$$(2-\lambda)\Delta + 2\lambda y - 3(2-\lambda)B(y,\Delta) = \delta E\left[\frac{\partial A}{\partial y}(z,\Delta') + \lambda B(z,\Delta') - H'\frac{\partial B}{\partial y}(z,\Delta')\right], \tag{E.7}$$

$$(2-\lambda)A(y,\Delta) = 2 + \delta E\left[\frac{\partial B}{\partial y}(z,\Delta') - \lambda A(z,\Delta') - H'\frac{\partial A}{\partial y}(z,\Delta')\right], \tag{E.8}$$

where $H' := \lambda z + (2-\lambda)z'$.

Lemma 5. *Every interior symmetric MPE in polynomial strategies is either affine or quadratic. More precisely, there exist constants* $m, a, \nu \in \mathbb{R}$ *and* $b \in (0,1)$ *such that*

$$P_0(y,\Delta) = m + a\Delta + by + \nu z(y,\Delta)^2, \qquad P_1(y,\Delta) = m - a\Delta - by + \nu z(y,\Delta)^2, \tag{E.9}$$

where

$$z(y,\Delta) = \Big(\frac{1}{2} - a\Big)\Delta - by. \tag{E.10}$$

In particular, every odd monomial of total degree at least 3 *and every even monomial of total degree at least* 4 *must vanish.*

The coefficients satisfy

$$4\lambda a = 2\lambda - (2-\lambda)b, \tag{E.11}$$

$$2\delta\nu b^3 + (2-\lambda)\delta b^3 - 3(2-\lambda)b + 2\lambda = 0, \tag{E.12}$$

$$\nu\Big((2-\lambda) + 3\delta\lambda b^2 - 2\delta(2-\lambda)b^3\Big) = 0, \tag{E.13}$$

$$m\big(2-\lambda(1-\delta)\big) = 2 + \delta b - \delta\nu\Big(\frac{1}{2}-a\Big)^2\big[\lambda - 2(2-\lambda)b\big]\sigma^2. \tag{E.14}$$

Hence there are at most two possible interior symmetric MPEs in polynomial strategies:

- ***Affine:*** $\nu = 0$*, in which case*

$$(2-\lambda)\delta b^3 - 3(2-\lambda)b + 2\lambda = 0, \qquad a = \frac{2-\delta b^2}{2(3-\delta b^2)}, \qquad m = \frac{2+\delta b}{2-\lambda(1-\delta)}.$$

- ***Quadratic:*** $\nu \neq 0$*, in which case*

$$(2-\lambda) + 3\delta\lambda b^2 - 2\delta(2-\lambda)b^3 = 0,$$

$$\nu = \frac{(2-\lambda)b(3-\delta b^2) - 2\lambda}{2\delta b^3}, \qquad a = \frac{2\lambda - (2-\lambda)b}{4\lambda},$$

  with m *given by* (E.14)*.*

If $\lambda \geq 4/5$*, no interior symmetric MPE in quadratic strategies exists.*

Proof. The proof has three steps.

Step 1: Market shares must follow an affine law of motion.

Define firm 0's continuation value

$$\Phi(u) := E\left[\lambda P_0(u,\Delta') + \big(1 + \lambda u + (2-\lambda)\tau(u,\Delta')\big)\frac{\partial P_1}{\partial y}(u,\Delta')\right],$$

where

$$\tau(u,\Delta') := \frac{\Delta' + P_1(u,\Delta') - P_0(u,\Delta')}{2}.$$

Because P_0 and P_1 are polynomials, so is τ. Because Δ' has bounded support and the integrand is polynomial in (u,Δ'), Φ is itself a polynomial in u.

Rearranging (E.5), there exists a one-variable polynomial F such that

$$P_0(y,\Delta) = \frac{2}{2-\lambda} + \frac{2\lambda}{2-\lambda}y + F(z(y,\Delta)). \tag{E.15}$$

Applying symmetry (E.1) to the definition of z (E.2) gives

$$z(-y,-\Delta) = -z(y,\Delta).$$

Then, applying symmetry to the left-hand side of (E.15) yields

$$P_1(y,\Delta) = P_0(-y,-\Delta) = \frac{2}{2-\lambda} - \frac{2\lambda}{2-\lambda}y + F(-z(y,\Delta)). \tag{E.16}$$

Subtracting expressions for (E.15) and (E.16) gives

$$P_0(y,\Delta) - P_1(y,\Delta) = \frac{4\lambda}{2-\lambda}y + F(z) - F(-z).$$

But by the definition of z (E.2),

$$P_0(y,\Delta) - P_1(y,\Delta) = \Delta - 2z(y,\Delta).$$

Let $F_o(z) := \frac{F(z)-F(-z)}{2}$ denote the odd part of F. Then

$$\frac{\Delta}{2} - \frac{2\lambda}{2-\lambda}y = z(y,\Delta) + F_o(z(y,\Delta)). \tag{E.17}$$

Set $L(u) := u + F_o(u)$. The left-hand side of (E.17) is a non-constant affine polynomial (degree 1), while the right-hand side is the composition $L \circ z$. By the degree of composition rule for polynomials, the right-hand side has degree

$$\deg(L \circ z) = \deg(L)\deg(z),$$

and therefore $\deg L = \deg z = 1$ (i.e., L is affine). Since L is the sum of odd functions u and $F_o(u)$, it is also odd. Since every affine and odd function is linear, we must have $L(u) = \kappa u$, where $\kappa \neq 0$ since L has degree 1. Moreover,

$$\kappa z(y,\Delta) = \frac{\Delta}{2} - \frac{2\lambda}{2-\lambda}y, \tag{E.18}$$

so z is affine. Without loss, we can write the affine function z as

$$z(y,\Delta) = \Big(\frac{1}{2} - a\Big)\Delta - by \tag{E.19}$$

for some constants $a, b \in \mathbb{R}$. Matching coefficients in (E.18), gives $\kappa\Big(\frac{1}{2} - a\Big) = \frac{1}{2}$ and $\kappa b = \frac{2\lambda}{2-\lambda} > 0$. Eliminating $\kappa \neq 0$ yields

$$4\lambda a = 2\lambda - (2-\lambda)b, \tag{E.20}$$

establishing (E.11).

By assumption, the equilibrium is interior, so $z\Big(\pm\frac{1}{2}, 0\Big) \in (-\frac{1}{2}, \frac{1}{2})$. Substituting $(\pm\frac{1}{2}, 0)$ in (E.19), gives $z\Big(\pm\frac{1}{2}, 0\Big) = \mp\frac{b}{2} \in (-\frac{1}{2}, \frac{1}{2})$, so

$$|b| < 1. \tag{E.21}$$

Moreover $b \neq 0$ (since $\kappa b = \frac{2\lambda}{2-\lambda} > 0$).

Finally, since

$$P_0 - P_1 = \Delta - 2z = 2(a\Delta + by),$$

and the common part $A = (P_0 + P_1)/2$ is even, there exist a constant m and an even one-variable polynomial C such that

$$P_0(y,\Delta) = m + a\Delta + by + C(z(y,\Delta)), \qquad P_1(y,\Delta) = m - a\Delta - by + C(z(y,\Delta)). \tag{E.22}$$

Thus any nonlinearity in prices is subsumed by the same even polynomial C evaluated at the affine state transition z.

Step 2: The even part has degree at most 2.

Substitute (E.22) and (E.19) into the difference equation (E.7). Since

$$B(y,\Delta) = a\Delta + by, \qquad A(y,\Delta) = m + C(z),$$

we have

$$\frac{\partial B}{\partial y}(z,\Delta') = b, \qquad \frac{\partial A}{\partial y}(z,\Delta') = -bC'(z'),$$

where $z' = \left(\frac{1}{2} - a\right)\Delta' - bz$. Also,

$$B(z,\Delta') = a\Delta' + bz = \frac{\Delta'}{2} - z'.$$

Using $E[\Delta'] = 0$ and therefore $E[z'] = -bz$, equation (E.7) becomes

$$\big[(2-\lambda)(1-3a)\big]\Delta + \big[2\lambda - 3(2-\lambda)b\big]y = \delta\left\{(2-\lambda)b^2 z - b\,E[C'(z')]\right\}. \tag{E.23}$$

Express the even polynomial C as

$$C(u) = c_{2N}u^{2N} + c_{2N-2}u^{2N-2} + \cdots + c_0, \qquad c_{2N} \neq 0.$$

If $N \geq 2$, then

$$C'(u) = 2Nc_{2N}u^{2N-1} + \text{lower odd powers}.$$

Because

$$z' = \left(\frac{1}{2} - a\right)\Delta' - bz,$$

the coefficient of z^{2N-1} in $(z')^{2N-1}$ is $(-b)^{2N-1}$. Hence the coefficient of z^{2N-1} in $E[C'(z')]$ is $2Nc_{2N}(-b)^{2N-1}$, so the coefficient of z^{2N-1} on the right-hand side of (E.23) is

$$-\delta b \cdot 2Nc_{2N}(-b)^{2N-1} = 2N\delta c_{2N}b^{2N}.$$

This is nonzero because $c_{2N} \neq 0$ and $b \neq 0$. Therefore the right-hand side would have total degree $2N-1 > 1$ as a polynomial in (y,Δ), whereas the left-hand side of (E.23) has total degree 1, a contradiction. Hence $N \leq 1$, so C has degree at most 2.

Since C has degree at most 2, we can write it as

$$C(u) = \nu u^2 + c_0.$$

Without loss, we can absorb the constant c_0 into m in equation (E.22), so that $C(u) = \nu u^2$ and therefore

$$P_0(y,\Delta) = m + a\Delta + by + \nu z(y,\Delta)^2, \qquad P_1(y,\Delta) = m - a\Delta - by + \nu z(y,\Delta)^2. \tag{E.24}$$

Therefore, every interior symmetric MPE in polynomial strategies is at most quadratic.

Step 3: Coefficients in the two cases.

With $C(u) = \nu u^2$, we have $C'(u) = 2\nu u$, so

$$E[C'(z')] = 2\nu E[z'] = -2\nu bz,$$

where we used $z' = \left(\frac{1}{2} - a\right)\Delta' - bz$ and $E[\Delta'] = 0$. Hence (E.23) becomes

$$\big[(2-\lambda)(1-3a)\big]\Delta + \big[2\lambda - 3(2-\lambda)b\big]y = \delta\big[(2-\lambda) + 2\nu\big]b^2 z. \tag{E.25}$$

Using $z = ((1/2) - a)\Delta - by$, coefficient matching gives

$$(2-\lambda)(1-3a) = \delta\big[(2-\lambda)+2\nu\big]b^2\Big(\frac{1}{2}-a\Big), \tag{E.26}$$

$$2\lambda - 3(2-\lambda)b = -\delta\big[(2-\lambda)+2\nu\big]b^3. \tag{E.27}$$

Equation (E.27) can be rearranged as (E.12).

Next consider the sum equation (E.8). Substituting $A(y,\Delta) = m + \nu z^2$, $\frac{\partial A}{\partial y}(z,\Delta') = -2\nu b z'$, and $\frac{\partial B}{\partial y}(z,\Delta') = b$ into (E.8) yields

$$(2-\lambda)(m+\nu z^2) = 2 + \delta\Bigg[b - \lambda\big(m + \nu E[z'^2]\big) + 2\nu b E[H'z']\Bigg].$$

Using $E[z'^2] = \Big(\frac{1}{2}-a\Big)^2\sigma^2 + b^2 z^2$ and

$$\begin{aligned} E[H'z'] &= E\big[(\lambda z + (2-\lambda)z')z'\big] \\ &= \big[-\lambda b + (2-\lambda)b^2\big]z^2 + (2-\lambda)\Big(\frac{1}{2}-a\Big)^2\sigma^2, \end{aligned}$$

we obtain

$$(2-\lambda)(m+\nu z^2) = 2 + \delta b - \delta\lambda\left[m + \nu\Big(\frac{1}{2}-a\Big)^2\sigma^2\right] + 2\nu b\delta(2-\lambda)\Big(\frac{1}{2}-a\Big)^2\sigma^2 + \delta\nu\left[2(2-\lambda)b^3 - 3\lambda b^2\right]z^2$$

Matching coefficients of z^2 and the constant term gives

$$\nu\Big((2-\lambda) + 3\delta\lambda b^2 - 2\delta(2-\lambda)b^3\Big) = 0, \tag{E.28}$$

and

$$m\big(2-\lambda(1-\delta)\big) = 2 + \delta b - \delta\nu\Big(\frac{1}{2}-a\Big)^2\big[\lambda - 2(2-\lambda)b\big]\sigma^2, \tag{E.29}$$

establishing equations (E.13) and (E.14).

If $\nu = 0$, then (E.12) reduces to

$$(2-\lambda)\delta b^3 - 3(2-\lambda)b + 2\lambda = 0.$$

Because of (E.21), any admissible affine equilibrium must have $b \in (-1,1)$. On $[-1,1]$, the cubic function

$$f(b) := (2-\lambda)\delta b^3 - 3(2-\lambda)b + 2\lambda$$

is strictly decreasing, and satisfies $f(0) = 2\lambda > 0$ and $f(1) = (2-\lambda)(\delta-3) + 2\lambda < 0$. Hence the only admissible root lies in $(0,1)$. Combining the cubic with (E.11) gives

$$a = \frac{2-\delta b^2}{2(3-\delta b^2)}, \qquad m = \frac{2+\delta b}{2-\lambda(1-\delta)}.$$

As expected, this coincides with Proposition 1 as strategies are affine when $\nu = 0$.

If instead $\nu \neq 0$, then (E.13) implies

$$(2-\lambda) + 3\delta\lambda b^2 - 2\delta(2-\lambda)b^3 = 0, \tag{E.30}$$

while (E.12) implies

$$\nu = \frac{(2-\lambda)(3b-\delta b^3)-2\lambda}{2\delta b^3}.$$

This is the quadratic coefficient. Rewriting (E.11) gives

$$a = \frac{2\lambda-(2-\lambda)b}{4\lambda}.$$

We claim that $b \in (0,1)$. Note first that $b < 1$ by (E.21). Rearranging equation (E.30) (using $b \neq 0$), gives

$$2(2-\lambda)b - 3\lambda = \frac{2-\lambda}{\delta b^2} > 0$$

and therefore

$$b > \frac{3}{2}\frac{\lambda}{2-\lambda} > 0. \tag{E.31}$$

Since $b < 1$ by equation (E.21), we find

$$\frac{3}{2}\frac{\lambda}{2-\lambda} < b < 1 \therefore \lambda < \frac{4}{5},$$

showing that the quadratic equilibrium does not exist if $\lambda \geq 4/5$.

□

Lemma 5 shows that any interior symmetric MPE in polynomial strategies must be in one of the two cases above. While it does not prove that the MPE in quadratic strategies exists, it determines prices and market shares when it does. In particular, since both affine and quadratic MPE satisfy equation (E.10) with $b \in (0,1)$, the result from Proposition 2 generalizes:

Proposition 7. *Fix an interior symmetric MPE in polynomial strategies. Then, market shares follow a stable, oscillating AR(1) process:*

$$y_{t+1} = -by_t + \Big(\frac{1}{2}-a\Big)\Delta_t,$$

where $b \in (0,1)$.

The no-regulation benchmark has a unique MPE:

$$p_0^U(\Delta) = c - \frac{\delta\lambda}{2-\lambda}K + \frac{c}{3}\Delta, \qquad p_1^U(\Delta) = c - \frac{\delta\lambda}{2-\lambda}K - \frac{c}{3}\Delta. \tag{E.32}$$

Hence the average unregulated price is the constant:

$$\bar{p}_U(\Delta) := \frac{p_0^U(\Delta)+p_1^U(\Delta)}{2} = c\left(1 - \frac{\delta\lambda}{2-\lambda}\frac{K}{c}\right). \tag{E.33}$$

By Lemma 5, in any symmetric interior polynomial MPE, the cross-firm average price under regulation is

$$\bar{p}_R(y,\Delta) := \frac{p_0(y,\Delta)+p_1(y,\Delta)}{2} = c\left[m + \nu z(y,\Delta)^2\right]. \tag{E.34}$$

Proposition 8. *Fix an interior symmetric MPE in polynomial strategies. For every state* (y,Δ),

$$\bar{p}_R(y,\Delta) > \bar{p}_U(\Delta).$$

Proof. By Lemma 5, any such MPE is either affine or quadratic. The affine case has been shown in Proposition 3 in the text.

Consider the quadratic case. By Lemma 5, the formula for ν is

$$\nu = \frac{(2-\lambda)b(3-\delta b^2) - 2\lambda}{2\delta b^3}. \tag{E.35}$$

The denominator is positive. Because $b \in (0,1)$ and $\delta < 1$, we have $3 - \delta b^2 > 2$. Therefore

$$(2-\lambda)b(3-\delta b^2) - 2\lambda > 2(2-\lambda)b - 2\lambda > 0,$$

where the last inequality uses (E.31). Hence, the numerator is also positive, so $\nu > 0$.

Now use the formula for m:

$$m\big(2-\lambda(1-\delta)\big) = 2 + \delta b - \delta\nu\Big(\frac{1}{2} - a\Big)^2 \big[\lambda - 2(2-\lambda)b\big]\sigma^2.$$

Since $\lambda - 2(2-\lambda)b < 0$ by equation (E.31), and $\nu > 0$, the last term on the RHS is strictly positive. Hence,

$$m\,[2-\lambda(1-\delta)] > 2 + \delta b > 2.$$

Since $2 - \lambda(1-\delta) < 2$, we conclude that $m > 1$. Together with $\nu > 0$ and $z(y,\Delta)^2 \geq 0$, this implies

$$\bar{p}_R(y,\Delta) = c\left[m + \nu z(y,\Delta)^2\right] > c \qquad \text{for every } (y,\Delta).$$

By equation (E.33),

$$\bar{p}_U = c - \frac{\delta\lambda}{2-\lambda}K < c.$$

Therefore, $\bar{p}_U(\Delta) < c < \bar{p}_R(\Delta, y)$ for all Δ, y.

$\square$